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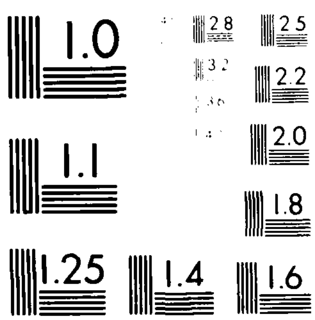

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A STOCHASTIC THEORY OF PRODUCTION CORRESPONDENCES.

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A STOCHASTIC THEORY OF PRODUCTION CORRESPONDENCES

by

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Operations Research Center
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MARCH 1981 81-6

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ABSTRACT

An axiomatic model of production technology under uncertainty is formulated from a purely technical point of view as a generalization of Shephard [1970-a] and Shephard/Färe [1980]'s framework for a general theory of production. The uncertain technical feasibility of production is characterized by inversely related stochastic input and output correspondences. This model of technology is then synthesized with a Radner [1968] type model of information for a formulation of production policies. This synthesis is used to give formulations of laws of returns under uncertainty. Finally, a generalized notion of homothetic production correspondences is developed to give special-structured stochastic production models which allow explicit consideration of optimal production policies.

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INTRODUCTION

In economic theory, a model of production is a mathematical system which characterizes the technical relationship between the outputs of a production system and the inputs of the factors of production. Traditionally, production is modelled by production functions, the chief purpose of which is to display the possibility of substitution between the factors of production to achieve a certain output. However, in actual production, it is common to have multiple products with possibilities of substitution between them. Furthermore, many production processes are dynamic in nature, and greatly influenced by unforseen forces of the environment of production. In such cases, the framework of production functions is woefully inadequate as a model of production. Shephard [1970-a] and Jacobsen [1970] developed a theory of production correspondences to model steady state production systems with multiple products. Recently, Krug [1976] gave a model of stochastic production correspondences; and Shephard/Färe [1980] extended the framework of production correspondences to model dynamic production systems. Building upon these development, it is the purpose of this paper to develop a framework for the modelling of dynamic production under uncertainty as a further step towards a general theory of production.

In economic literature, there had been much emphasis that a production model is relevant only to a particular production unit; that the capital stock should be carefully distinguished from the flow commodities; that the long-run production function is fundamentally different from the short-run production function; that free goods should be excluded from the model; and the information concerning the production environment

available to the producer plays a role in the structure of the production model. But, ideally, a production model is a collection of statements characterizing the purely technical alternatives under various environments of production without regard to their execution!

The viewpoint taken in this paper and described below was first expounded definitively in Shephard [1967] : - Neither the exclusion of free goods, nor the requirement that the production model expresses the variable, substitutional, consummable character or the limitational, fixed stock character of the productive factors, nor the information and organization structures, as qualifications peculiar to a particular production unit, are logically necessary for the formulation of a production model.

A production model is a mathematical construction describing some well defined production technology. This production technology consists of a family of conceivable engineering arrangements, possibly over time, which are feasible under appropriate production environments. This family is not restricted necessarily to particular arrangements realized in practice. It possibly spans over historical changes and adaptations to the changing environment. Once defined, the technology implies a certain set of factors of production and outputs. No limitation will be put upon the availability of the productive factors. Thus, the production model will be taken to describe the unconstrained technical possibilities of production without being limited to any existing or planned production unit.

If a production model is to characterize purely technical possibilities, the available or projected means of a production unit and its

organization/information structure are not relevant. Such a particular unit merely prescribes a particular realization of the technology which may be considered by imposing constraints on the input and output flows and the choice of production programs compatible with the information available to the production unit.

In the following chapters, the forgoing conceptions of a model of production in stochastic terms are developed in some detail as a generalization of Shephard [1970-a] and Shephard/Färe [1980]. Chapter 1 gives an axiomatic formulation of a stochastic production technology. It also discusses, in a general setting, the information aspects of production and the notion of production policies. The synthesis of the notions of technology and information in Chapter 2 gives some formulations of laws of returns under uncertainty. Chapter 3 uses a generalized notion of homotheticity to give special-structured stochastic models of production whose simplicity enables explicit formulations of production policies.

CHAPTER 1

MODEL OF THE TECHNOLOGY

1.1 Framework of the Model

The model of production proposed in this paper characterizes a well defined technology by stating all the technically feasible alternatives of transforming factors of production (inputs) into *net* outputs under the various environments of production whose exact realization is possibly not forseen (i.e., there may be uncertainty about the environment).

The environment of production is described in terms of environmental variables which are not controlled by any producer. Following the terminology of statistical decision theory, a complete specification of the production environment is called a *state of nature*. A state of nature, or simply a state, is a complete description of the production environment from the beginning to the end of the production processes being studied. The collection of all the possible states of nature is called a *state space*, and is denoted by S . The state space is taken to be a probability space $(S,\mathcal{S},\mathcal{P})$ with σ-algebra $\mathcal{S}$ and probability measure $\mathcal{P}$. An element of the σ-algebra $\mathcal{S}$ describing some *aspect* of the environment is called an *event* . It is tacitly assumed that $\mathcal{P}$ is an objective probability measure on the events of the environment as prescribed by the statistical laws of nature.

$\mathcal{S}$ will be assumed to be the finest σ-algebra that is ever distinguishable by any producer. This means if $(S,\bar{\mathcal{S}},\bar{\mathcal{P}})$ represents the subjective assessment of the likelihood of the events of nature by a producer, the σ-algebra $\bar{\mathcal{S}}$ is a sub-algebra of $\mathcal{S}$. Of course, a

producer is entitled to have his own beliefs, hence $\bar{\mathcal{P}}$ need not be a restriction of $\mathcal{P}$ on the sub-algebra $\bar{\mathcal{B}}$.

In an abstract model of production, there is no need to impose further conditions on the state space $(S,\mathcal{B},\mathcal{P})$. However, in applications, quantitative descriptions of the environment will usually require S to be a metric space. For mathematical expediance, it will be assumed that $(S,\mathcal{B},\mathcal{P})$ is a complete probability space.

The collection of inputs relevant to a technology being modelled will be denoted generically by X , and called an *input space*. In order for a space X to be meaningfully defined as an input space, every element in it must be "non-negative" and has a measure of "size". Furthermore, scalar multiplication and addition must be well defined on X ; and X should be complete in some sense. So, an input space X will be taken to be the non-negative orthant of a complete normed vector space. Similar reasoning applies to the definition of an *output space*, generically denoted by U . Once specified, the input and output spaces dictate that a certain set of goods and services as input factors, and another set of goods and serivces as net outputs. It is assumed that both these sets are finite and their cardinality are denoted by n and m respectively.

The following pairs of input and output spaces X and U are frequently used:

(a) For single period or steady state production models: $\mathbb{R}^n_+$ & $\mathbb{R}^m_+$.

(b) For finite (T) period production models: $(l^T_\infty)^n_+$ & $(l^T_\infty)^m_+$.

(c) For infinite period production models: $(l_\infty)^n_+$ & $(l_\infty)^m_+$.

(d) For infinite horizon continuous time production models:

$$X \equiv (L_\infty(\mathbb{R}_+,\Sigma_1,\mu_1))_+ \times \dots \times (L_\infty(\mathbb{R}_+,\Sigma_n,\mu_n))_+ =: (L_\infty)_+^n$$

$$\& \quad U \equiv (L_\infty(\mathbb{R}_+,\Gamma_1,\rho_1))_+ \times \dots \times (L_\infty(\mathbb{R}_+,\Gamma_m,\rho_m))_+ =: (L_\infty)_+^m$$

where $L_\infty(\mathbb{R}_+,\Sigma_i,\mu_i)$ and $L_\infty(\mathbb{R}_+,\Gamma_j,\rho_j)$ are Lebesgue spaces with $(\mathbb{R}_+,\Sigma_i,\mu_i)$ and $(\mathbb{R}_+,\Gamma_j,\rho_j)$ being positive σ-finite measure spaces (on the time axis $\mathbb{R}_+$). Denoting $\|f_i\|$ as the norm of $f_i \in L_\infty(\mathbb{R}_+,\Sigma_i,\mu_i)$, the norm for the product space $(L_\infty)_+^n$ may be taken as: $\|f\| := \operatorname{Max}_i \|f_i\|$ for $f \equiv (f_1, \dots, f_n) \in (L_\infty)_+^n$.

The above input and output spaces are the non-negative orthants of well-known Banach spaces $\mathbb{R}^n$, $(l_\infty)^n$ and $(L_\infty)^n$ etc. whose definitions and properties may be found in Dunford/Schwartz [1957]. Note that in cases (b), (c) and (d), the inputs and outputs are functions (in time). Occasionally, they are referred to as *input histories* and *output histories* in order to stress their dynamic nature. The spaces in (a) are used in Shephard [1970-a]; those of (d) are used in Shephard/Färe [1980]; and (b), (c) are discrete-time versions of (d).

Since X and U in general are the non-negative orthant of product Banach spaces, the meaning of their null element 0 should be unambiguous. Furthermore, with the standard representation of vectors $x \equiv (x_1, \dots, x_n)$, $x \in X$ and $u \equiv (u_1, \dots, u_m)$, $u \in U$; the meaning of the usual partial ordering on vector spaces, namely $\geqq$, $\geq$ and $>$, should be clear. For a more detailed exposition of these notations, see Shephard/Färe [1980].

With the above preliminaries, a stochastic production technology may now be formally defined as:

<u>(1.1.1) Definition</u>: A *stochastic production technology*, or more specifically a *stochastic technical feasibility set*, $\mathcal{T}$ is a subset of the product space $X \times S \times U$ such that $(x,s,u) \in \mathcal{T}$ if and only if output u is attainable with the input x under state s.

1.2 The Output Correspondences

A stochastic technology $\mathcal{T}$ may be represented in various ways. As a useful representation, output correspondences are defined herewith:

<u>(1.2.1) Definition</u>: A *stochastic output correspondence* P of a stochastic technology $\mathcal{T}$ is a correspondence $P : X \times S \to 2(U)$ defined by:

$$(x,s) \in X \times S \longmapsto P(x,s) := \{u \in U \mid (x,s,u) \in \mathcal{T}\} .$$

where $2(U)$ is the power set of the space U.

Clearly, the set $P(x,s)$ is the collection of all the outputs attainable with input x under state s. It is convenient to define two restricted correspondences as follows:

(1.2.2) For each $x \in X$, $s \in S \longmapsto P_x(s) := P(x,s)$.

(1.2.3) For each $s \in S$, $x \in X \longmapsto P_s(x) := P(x,s)$.

1.3 Axioms on the Technology

A stochastic model of a production technology is now completed by specification of properties. This will be done by stating a set of axioms which are imposed on its associated output correspondence P (and P_x ; P_s). For this purpose, first define:

<u>(1.3.1) Definition</u>: A correspondence H from a measurable space $(S,\mathcal{S})$ to a topological space U is said to be *measurable* if the inverse image

$$H^{-1}(F) := \{s \in S \mid H(s) \cap F \neq \emptyset\}$$

belongs to $\mathcal{S}$ for every closed set F in U.

The use of closed sets in the above definition is convenient since every singleton $\{u\}$ in U is a closed set.

The following properties (some of which are stated with various strength) are to be taken as axioms on the output correspondence of a technology:

<u>Measurability</u>

P0 For each $x \in X$, the correspondence P_x (see (1.2.2)) is measurable.

<u>Nothing from Nothing</u>

P1 For each state $s \in S$, $P(x = 0, s) = \{0\}$. The null output 0 belongs to $P(x,s)$ for all $x \in X$ and $s \in S$.

<u>Bounds on Outputs</u>

P2 For each state $s \in S$ and input $x \in X$, $P(x,s)$ is bounded; i.e., there exists $B \in (0,+\infty)$ such that the norm $\|u\| \leq B$ for all $u \in P(x,s)$.

P2.I For each $x \in X$, the correspondence P_x is integrably bounded; i.e., there exists a non-negative integrable function $g : S \to \mathbb{R}_+$ such that for each output $u \in U$ and state $s \in S$, $u \in P(x,s)$

implies $\|u\| \leq g(s)$.

P2.S For each state $s \in S$ and input $x \in X$, $P(x,s)$ is totally bounded.

Disposability of Inputs

P3 For each $s \in S$ and $x \in X$, $P(x,s) \subset P(\lambda \cdot x,s)$ if $\lambda \geq 1$.

P3.S For each $s \in S$ and $x \in X$, $y \geq x$ implies $P(x,s) \subset P(y,s)$.

Attainability of Outputs

P4.1 For each $i \in \{1, \ldots, m\}$ there is an output $u \in U$ with $u_i \neq 0$, an input $x \in X$ and an event $A \in \mathcal{B}$ with $\mathcal{P}(A) > 0$ such that $u \in P(x,s)$ for each $s \in A$.

P4.2 Suppose an output $u \neq 0$ and $u \in P(x,s)$. Then for each positive scaling factor θ , there is a positive scalar λ (depending on x , u and s) such that $\theta \cdot u \in P(\lambda \cdot x,s)$.

P4.2.I For every input $x \in X$ and positive scaling factor θ , there is a function $\lambda_{\theta,x} : S \to \mathbb{R}_{++}$ such that for every output $u \in P(x,s)$, $\theta \cdot u \in P(\lambda_{\theta,x}(s) \cdot x,s)$; and the function $\lambda_{\theta,x}$ is integrable.

Closure and Continuity Property

P5 For each state $s \in S$, the graph of the correspondence P_s (see (1.2.3)) is closed; i.e., $x^k \to x^o$, $u^k \to u^o$ and $u^k \in P(x^k,s)$ for all k implies $u^o \in P(x^o,s)$.

P5.C For each state $s \in S$, the correspondence P_s is upper-hemi-continuous (u.h.c.); i.e., for each $\bar{x} \in X$ and every open neighborhood G of $P(\bar{x},s)$ there is a neighborhood Z of $\bar{x}$ such that $P(x,s) \subset G$ for every $x \in Z$.

Disposability of Outputs

P6 For each state $s \in S$ and $x \in X$, if $u \in P(x,s)$ and $\theta \in [0,1]$, then $\theta \cdot u \in P(x,s)$.

P6.S For each state $s \in S$ and $x \in X$, if $u \in P(x,s)$ and $u \geqq v \in U$, then $v \in P(x,s)$.

The above set of axioms is a direct stochastic extension of those given in Shephard [1970-a], Shephard [1974] and Shephard/Färe [1980]. The readers are referred to them for a discussion of the economic meaning of P1, P2, P2.S, P3, P3.S, P4.1, P4.2, P6 and P6.S.

Particular to a stochastic model of production, Axiom P0 guarantees that for every closed set F in the output space, it is meaningful to speak of the probability that outputs in F are attainable. Axiom P2.I gives a uniform (over the states of nature) boundedness condition on the output sets, while Axiom P4.2.I gives a uniform scaling condition on the attainability of outputs.

Axiom P5 on the closure of the graph of P_s $(s \in S)$ is essentially a technical assumption. In particular, it guarantees that the output set $P(x,s)$ is closed for all $x \in X$ and $s \in S$. Whatever impression of continuity P5 conveys is formalized by P5.C, the upper-hemi-continuity of P_s. This notion of continuity is quite useful in establishing some interesting propositions in later sections. Otherwise, P5 is sufficient for most models of technology.

(1.3.2) Remark: Notice that some of the axioms stated are concerned with the vector properties of the output correspondence P while others are topological in nature. Since both the input space X and the output space U are taken to be subsets of Banach spaces, they have

the natural norm topology. However, sometimes a weaker topology is desirable. For example, when $U \equiv (L_\infty)^m_+$ the weak* topology on U may be more convenient for application (e.g., the boundedness (P2) of an output set $P(x,s)$ implies it is totally-bounded (P2.S) under the weak* topology). See Shephard/Färe [1980] for a construction of the weak* topology on $(L_\infty)^n_+$ and the price interpretation of its dual space.

(1.3.3) Remark: If there is only one state of nature (i.e., there can be no uncertainty concerning the realization of the production environment), then Axiom P0 is superfluous. The axiom structure {P1, P2, P3, P4.1, P4.2, P5 & P6} reduces naturally to the production model formulated in Shephard [1974] and Shephard/Färe [1980] with the appropriate choice of the input and output spaces.

1.4 Freedom of Axioms from Contradiction and their Independence

An axiom system as a model for production technology is free from contradiction if there is a technology which satisfies all the axioms in the system. Examples of stochastic output correspondences $P : X \times S \to 2^{(U)}$ are given below. They are based on widely used deterministic models of production. As a by-product, the axioms given in the last section are shown to be free from contradiction.

(1.4.1) Example: Cobb-Douglas production structure with random disturbances.

The following production model was used by Schmidt/Lovell [1979] in their estimation of the technical and allocative inefficiency of U.S. steam-electric generating plants:

A state $s \in S \equiv (-\infty,+\infty)$ is given as the sum of two terms:

$s := \gamma + \beta$. The random variable γ measures the random disturbances due to nature and is normally distributed, $N(0,\sigma_\gamma^2)$. The random variable $-\beta$ measures technical inefficiency. It is nonnegative and half-normal; i.e., it is the absolute value of a random variable normally distributed, $N(0,\sigma_\beta^2)$. Clearly, the Borel field of the real line may be taken as the σ-algebra $\mathcal{B}$.

The input space is $X \equiv \mathbb{R}_+^n$ and the output space is $U \equiv \mathbb{R}_+$. The output correspondence $P : X \times S \to 2(U)$ is defined by

$$(x,s) \in \mathbb{R}_+^n \times (-\infty,+\infty) \mapsto P(x,s) := \left\{ u \in \mathbb{R}_+ \;\middle|\; \begin{array}{l} u = \theta \cdot q \ , \ \theta \in [0,1] \\ q = Ae^s \prod_{i=1}^{n} x_i^{\alpha_i} \end{array} \right\}$$

where A and α_i's are positive constants, $\Sigma\alpha_i = 1$; and e is the exponential constant.

It is straight forward to verify that P satisfies Axioms P0 to P6. For details, see Appendix item (1.9.1) □

<u>(1.4.2) Example: Linear Activity Analysis Model</u>

Let there be K (≥ 1) productive activities which employ in total n types of exogenous inputs and yield m types of products. Let the state space be $(S,\mathcal{B},\mathcal{P})$. For each states $s \in S$, the non-negative $m \times K$ matrix $B(s)$ and the non-negative $n \times K$ matrix $A(s)$ denote respectively the output and input coefficient matrix; $B_{jk}(s)$ and $A_{ik}(s)$ are the amount of the j-th output and the required i-th input for activity k operating with unit intensity under state s . Clearly, $X \equiv \mathbb{R}_+^n$ and $U \equiv \mathbb{R}_+^m$. The following assumptions are imposed:

(1.4.2.1) The matrix-valued functions $s \mapsto B(s)$ and $s \mapsto A(s)$ are measurable via the homeomorphism between their range spaces and $\mathbb{R}_+^{mK}$, $\mathbb{R}_+^{nK}$, respectively.

(1.4.2.2) Each unit activity employs some minimal inputs under all possible states. Formally, for each $k \in \{1, \ldots, K\}$ there is a scalar $\varepsilon_k > 0$ such that $\sum_{i=1}^{n} A_{ik}(s) \geqq \varepsilon_k$ for all $s \in S$.

(1.4.2.3) The unit activities produce only finite outputs under all possible states. Formally, for each $j \in \{1, \ldots, m\}$ there is a scalar $M_j < +\infty$ such that $\sum_{k=1}^{K} B_{jk}(s) \leqq M_j$ for all $s \in S$.

(1.4.2.4) Each activity produces some output, and each output is attainable. Formally, for each $k \in \{1, \ldots, K\}$, $\mathbb{P}\left\{s \in S \mid \sum_{j=1}^{m} B_{jk}(s) = 0\right\} < 1$; and for each $j \in \{1, \ldots, m\}$, $\mathbb{P}\left\{s \in S \mid \sum_{k=1}^{K} B_{jk}(s) = 0\right\} < 1$.

The output correspondence $P : \mathbb{R}_+^n \times S \to 2^{(\mathbb{R}_+^m)}$ is defined by:

$$P(x,s) := \{u \in \mathbb{R}_+^m \mid z \in \mathbb{R}_+^K \text{ , } A(s)\cdot z \leqq x \text{ , } B(s)\cdot z \geqq u\} \text{ ,}$$

where z denotes a vector of intensity of operation for the K activities.

Again, to verify that this production structure satisfies the axioms stated in the last section is straight forward; see Appendix item (1.9.2) □

Examples (1.4.1) and (1.4.2) are not dynamic production models. Example (1.4.2) may be easily generalized to a dynamic model. The following exmaple is a natural stochastic extension of the dynamic ship-building production function formulated in Shephard et al. [1977]:

(1.4.3) Example: Dynamic Linear Activity Analysis Model

Suppose the K production activities in (1.4.2) may be operative at all time periods, labelled $t = 1,2, \ldots$. Production in period t is contingent upon the state of nature prevailing in that period. The state space relevant for period t being a probability space $(S_t, \mathcal{S}_t, \mathcal{P}_t)$. Let the state space for the infinite horizon production technology be $(S, \mathcal{S}, \mathcal{P})$ where $S = \overset{\infty}{\underset{t=1}{\times}} S_t$ and $\mathcal{S}$ is the corresponding product σ-algebra and $\mathcal{P}$ the completion of the product measure.

The model here is different from (1.4.2) in that transfer of goods and services from one period to the next is allowed as intermediate products. Let a non-negative $m \times K$ matrix $C_t(s) \equiv C_t(s_t)$ be the intermediate product requirement coefficient matrix for period t under state s , $t = 1,2, \ldots$. A history of transferred product is denoted by $V \equiv (V^1, V^2, \ldots, V^t, \ldots)$, $V^t \in \mathbb{R}^m_+$ $(t = 1,2, \ldots)$. An exogenous input history is $x \equiv (w\ ;\ y^1, y^2, \ldots, y^t, \ldots)$ where $w \in \mathbb{R}^m_+$ is the initial endowment of intermediate products available for the commencement of production at period 1; $y^t \in \mathbb{R}^n_+$ $(t = 1,2, \ldots)$ denotes the exogenous inputs in period t . Similarly, an output history is denoted as $u \equiv (u^1, u^2, \ldots, u^t, \ldots)$, $u^t \in \mathbb{R}^m_+$ $(t = 1,2, \ldots)$. Thus, it is convenient to let the input space X be $\mathbb{R}^m_+ \times (l_\infty)^n_+$ and the output space be $(l_\infty)^m_+$. The norm of $x \equiv (w;y) \in \mathbb{R}^m_+ \times (l_\infty)^n_+$ may be taken as $\max(|w|, \|y\|)$ where $|w|$ is the Euclidean norm of w and $\|y\|$ the

$(l_\infty)_+^n$ norm of y .

The input and output coefficient matrices in period t are denoted by $A_t(s) \equiv A_t(s_t)$ and $B_t(s) \equiv B_t(s_t)$ respectively. It is assumed for all t , the matrix-valued functions $s_t \in S_t \mapsto B_t(s_t)$ and $s_t \in S_t \mapsto A_t(s_t)$ satisfy assumptions (1.4.2.1) to (1.4.2.4) where the scalars ε_k's and M_j's are constant over t . The function $s_t \in S_t \mapsto C_t(s_t)$ satisfies conditions analogous to (1.4.2.1) and (1.4.2.2).

The output correspondence $P : X \times S \to 2(U)$ is defined by:

$$(x \equiv (w;y)\ ,\ s) \longmapsto P(x,s) := \left\{ u \in (l_\infty)_+^m \;\middle|\; \begin{array}{l} \text{for } t = 1,2,\ \ldots\ ;\ z^t \in \mathbb{R}_+^K ; \\ A_t(s_t)\cdot z^t \leqq y^t\ ; \\ u^t + V^t \leqq B_t(s_t)\cdot z^t\ ; \\ C_t(s_t)\cdot z^t \leqq V^{t-1}\ ;\ V^o \equiv w\ . \end{array} \right\} .$$

Verification of the axioms is given in Appendix item (1.9.3) □

Examples (1.4.1), (1.4.2) and (1.4.3) establish the following:

<u>(1.4.4) Proposition</u>: Axioms {P0, P1, P2, P2.I, P2.S, P3, P3.S, P4.1, P4.2, P4.2.I, P5, P6, P6.S} as a system is free from contradiction; and every subsystem of it is, of course, also free from contradiction.

An axiom in an axiom system is said to be *independent* in the system if there is a case where it is not fulfilled while all other axioms in the system are satisfied.

<u>(1.4.5) Proposition</u>: The axiom system {P0, P2, P2.I, P3, P4.1, P4.2, P5, P6} contains only independent axioms.

<u>Proof</u>: Note that P1 is not included in the system since P4.2 and P2

implies P1. Axiom P4.2.I is not included because it is not difficult to show that P2.S, P5 and P4.2 together implies the first part of P4.2.I; and P4.2.I by itself is stronger than P4.2.

In the following, stochastic production technologies are defined such that exactly one axiom in the system fails. The construction follows closely that given in Shephard/Färe [1980]. Since only the logical relationship between the axioms is of concern, it suffices to let both the input and output spaces be Euclidean. As notation, let the line segment between two points y and z in an Euclidean space be

$$\langle y,z\rangle := \{\theta\cdot y + (1-\theta)\cdot z \mid \theta \in [0,1]\} .$$

The verification of the following is trivial and will be left out.

(1.4.5.1) <u>P0 fails</u>: $S \equiv [0,1]$ with Borel measure. $P : \mathbb{R}^n_+ \times S \to 2(\mathbb{R}^n_+)$:

$$(x,s) \longmapsto P(x,s) := \begin{cases} \langle 0,2x\rangle , & \text{if } s \text{ is irrational;} \\ \langle 0,x\rangle , & \text{if } s \text{ is rational.} \end{cases}$$

(1.4.5.2) <u>P2 fails</u>: $S \equiv [0,1]$ with Borel measure. $P : \mathbb{R}^n_+ \times S \to 2(\mathbb{R}^n_+)$:

$$(x,s) \longmapsto P(x,s) := \begin{cases} \{0\} , & \text{if } x = 0 ; \\ \mathbb{R}^n_+ , & \text{if } s = 1 \text{ and } x \neq 0 ; \\ \langle 0,x\rangle , & \text{if } x \neq 0 \text{ and } s \in [0,1) . \end{cases}$$

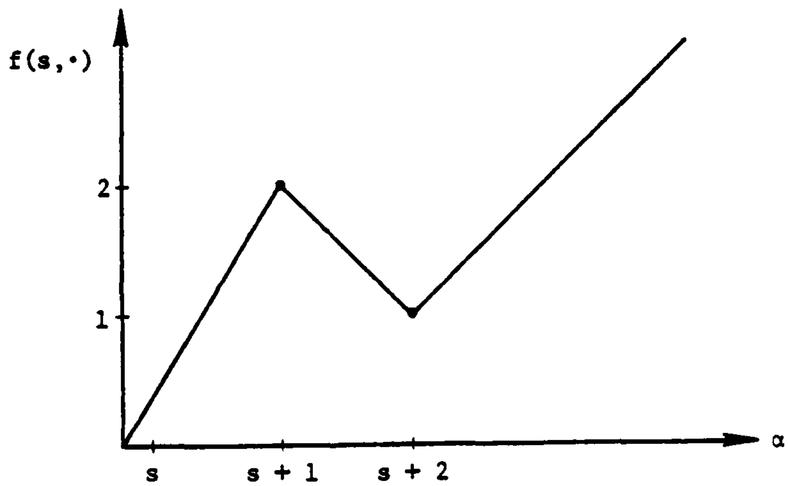
(1.4.5.3) <u>P2.I fails</u>: $S \equiv (0,1]$ with Borel measure. $P : \mathbb{R}^n_+ \times S \to 2(\mathbb{R}^n_+)$:

$$(x,s) \longmapsto P(x,s) := \langle 0,x/s\rangle .$$

(1.4.5.4) <u>P3 fails</u>: $S \equiv [0,1]$ with Borel measure. Suppose a function $f : S \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by:

$$(s,\alpha) \mapsto f(s,\alpha) := \begin{cases} 2\alpha/(1+s) & , \alpha \in [0\ ,\ s+1]\ ; \\ 3-\alpha+s & , \alpha \in (s+1\ ,\ s+2]\ ; \\ -1+\alpha-s & , \alpha \in (s+2\ ,\ +\infty)\ . \end{cases}$$

For a fixed $s \in S$, the graph of the function $f(s,\cdot)$ looks like:



$P : \mathbb{R}^n_+ \times S \to 2^{(\mathbb{R}^n_+)}$ is defined by $P(x,s) := \langle 0, f(s,\|x\|)\cdot x\rangle$.

(1.4.5.5) <u>P4.1 fails</u>: For each $\theta \in [0,\pi/2]$, let $v(\theta)$ be the vector in $\mathbb{R}^2_+$ with $\|v(\theta)\| = 1$ which substains an angle θ (in radian) with the first axis. $S \equiv [0,\pi/2]$ with $\mathcal{B}$ its Borel σ-algebra. For each event $A \in \mathcal{B}$, the probability measure is $\mathcal{P}(A) := 2\mu(A)/\pi$ where μ is the Borel measure on S. $P : \mathbb{R}^2_+ \times S \to 2^{(\mathbb{R}^2_+)}$ is defined by $P(x,s) := \langle 0, \|x\|\cdot v(s)\rangle$.

(1.4.5.6) <u>P4.2 fails</u>: $S \equiv [0,1]$ with Borel measure. $P : \mathbb{R}_+^n \times S \to 2(\mathbb{R}_+^n)$:

$$(x,s) \longmapsto P(x,s) := \left\langle 0, \beta(x,s) \cdot \frac{x}{\|x\|} \right\rangle ;$$

where $\beta(x,s) := \text{Min}\,\{\|x\| \ , \ s + 1\}$.

(1.4.5.7) <u>P5 fails</u>: $S \equiv [0,1]$ with Borel measure. $P : \mathbb{R}_+^n \times S \to 2(\mathbb{R}_+^n)$:

$$(x,s) \longmapsto P(x,s) := \begin{cases} \{0\} & , \text{ if } \|x\| \leqq 1 \ ; \\ \langle 0, s \cdot x \rangle & , \text{ if } \|x\| > 1 \ . \end{cases}$$

(1.4.5.8) <u>P6 fails</u>: $S \equiv [0,1]$ with Borel measure. $P : \mathbb{R}_+^n \times S \to 2(\mathbb{R}_+^n)$:

$$(x,s) \longmapsto P(x,s) := \begin{cases} \left\langle s \cdot \frac{x}{\|x\|} , x \right\rangle & , \text{ if } \|x\| \geqq s \ ; \\ \{0\} & , \text{ if otherwise. } \square \end{cases}$$

Following Shephard [1974], define

<u>(1.4.6) Definition</u>: The axiom system {P0, P1, P2, P3, P4.1, P4.2, P5 & P6} will be called the *stochastic weak axioms* for a stochastic production technology.

The system of stochastic weak axioms serves as a minimal core of the properties one would impose on a stochastic production technology. Stronger versions of the axioms are to be invoked only when the need arises.

<u>1.5 Input Correspondence and its Measurability</u>

As an alternative representation of a stochastic technology $\mathcal{T}$, define:

<u>(1.5.1) Definition</u>: *A stochastic input correspondence* L of a technology $\mathcal{T}$ is a correspondence $L : U \times S \to 2(X)$ defined by:

$$(u,s) \in U \times S \longrightarrow L(u,s) := \{x \in X \mid (x,s,u) \in \mathcal{T}\} .$$

Note that $(x,s,u) \in \mathcal{T} \iff u \in P(x,s) \iff x \in L(u,s)$. Thus P and L may be taken as inversely related representations of a stochastic technology:

$$L(u,s) \equiv \{x \in X \mid u \in P(x,s)\} \; ; \; P(x,s) \equiv \{u \in U \mid x \in L(u,s)\} .$$

It is again convenient to define two restricted correspondences as follows:

(1.5.2) For each $u \in U$, $s \in S \mapsto L_u(s) := L(u,s)$.

(1.5.3) For each $s \in S$, $u \in U \mapsto L_s(u) := L(u,s)$.

The inverse relationship between P and L allows the properties of L to be derived from the axioms on P stated in Section 1.3. They are as follows:

L1 For each state $s \in S$, $L(u = 0 , s) = X$; if $u \neq 0$, $0 \notin L(u,s)$.

L2 For each state $s \in S$ and each infinite sequence of outputs $\{u^k\}$ with $\|u^k\| \to +\infty$, $\bigcap_k L(u^k,s)$ is empty.

L2.I For each event $A \in \mathcal{B}$ with $\mathcal{P}(A) > 0$, $\|u^k\| \to +\infty$ implies $\bigcap_{s \in A} \bigcap_k L(u^k,s)$ is empty.

L2.S For each non-null subset of outputs V which is not totally bounded, $\bigcap_{u \in V} L(u,s)$ is empty for all $s \in S$.

L3 For each state $s \in S$, every $u \in U$, $x \in L(u,s)$ implies $\lambda \cdot x \in L(u,s)$ for all $\lambda \in [1,+\infty)$.

L3.S For each state $s \in S$, every $u \in U$, $x \in L(u,s)$ and $y \geqq x$ implies $y \in L(u,s)$.

L4.1 For each $i \in \{1, \ldots, m\}$ there is an output $u \in U$ with $u_i \neq 0$ and an event $A \in \mathcal{B}$ with $\mathcal{P}(A) > 0$ such that $\bigcap_{s \in A} L(u,s) \neq \emptyset$.

L4.2 For each $s \in S$ and each $u \in U$, if $x \in L(u,s)$ and $x \neq 0$, then $L(\theta \cdot u,s) \cap \{\lambda \cdot x \mid \lambda \in \mathbb{R}_+\}$ is not empty for all $\theta \in \mathbb{R}_{++}$.

L4.2.I For each fixed $\theta \in \mathbb{R}_{++}$, if $(x,s,u) \in X \times S \times U$ is such that $x \in L(u,s)$, let $g_{\theta,x}(s,u) := \text{Inf}\,\{\alpha \in \mathbb{R}_+ \mid \alpha \cdot x \in L(\theta \cdot u,s)\}$. For each $x \in X$, define a function $\lambda_{\theta,x} : S \to \mathbb{R}_+$ by $\lambda_{\theta,x}(s) := \text{Sup}_u\,\{g_{\theta,x}(s,u) \mid x \in L(u,s)\}$. The function $\lambda_{\theta,x}$ is integrable for each $x \in X$.

L5 For each state $s \in S$, the graph of the correspondence L_s (see (1.5.3)) is closed.

L6 For each $s \in S$ and each $u \in U$, $x \in L(u,s)$ implies $x \in L(\theta \cdot u,s)$ for all $\theta \in [0,1]$.

L6.S For each $s \in S$ and each $u \in U$, $x \in L(u,s)$ and $v \leqq u$ implies $x \in L(v,s)$.

To deduce the properties of the input correspondence L listed above from the axioms on P, the arguments needed parallel those given in Shephard/Färe [1980], with the exception of L2.I and L4.2.I; hence they will not be given here. However, L2.I is merely a restatement of P2.I, and L4.2.I gives an explicit form for the scaling function $\lambda_{\theta,x}$

of Axiom P4.2.I. Conversely, the axioms on P may be deduced from the corresponding properties on L . Hence, the properties listed above may be alternatively taken as axioms on a stochastic technology.

One certainly would like to have the following measurability property on L to complement Axiom P0.

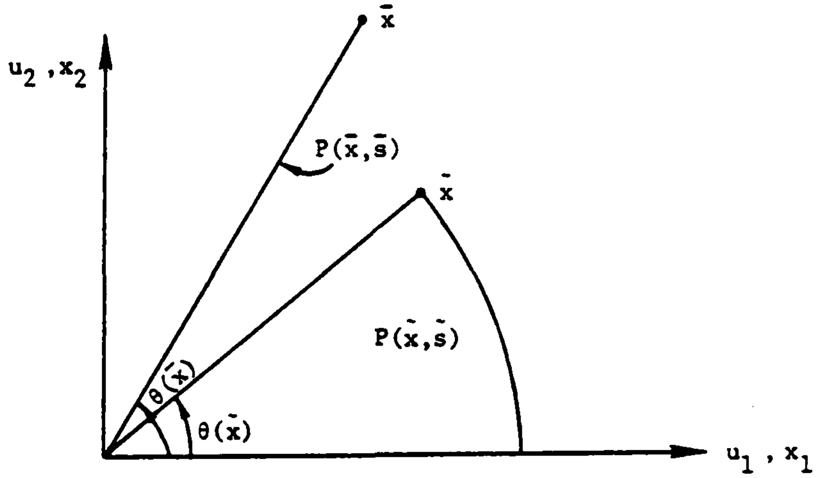
L0 For each $u \in U$, the correspondence L_u (see (1.5.2)) is measurable; i.e., for every closed subset F in X , $L_u^{-1}(F) := \{s \in S \mid L_u(s) \cap F \neq \emptyset\} \in \mathcal{S}$.

If L0 holds, then it is meaningful to speak of the event that a closed subset of inputs is sufficient to yield an output u . Unfortunately, L0 cannot be deduced from the stochastic weak axioms (1.4.6) on P . This is demonstrated by the following:

(1.5.4) Example: $S \equiv [0,\pi/2]$ with $\mathcal{S}$ being the Borel σ-algebra. The input space X is $\mathbb{R}_+^2$. For each $x \in \mathbb{R}_+^2$, $\theta(x)$ is defined to be the angle (in radian) substained by x with the x_1-axis. An output correspondence $P : \mathbb{R}_+^2 \times [0,\pi/2] \to 2^{(\mathbb{R}_+^2)}$ is defined by

$$P(x,s) := \begin{cases} \langle 0,x \rangle , & \text{if } s \text{ is rational or } \theta(x) \neq s ; \\ \{y \in \mathbb{R}_+^2 \mid \|y\| \leqq \|x\| , \theta(y) \leqq \theta(x)\} , & \text{if otherwise.} \end{cases}$$

Schematically, for inputs $\tilde{x}$, $\bar{x} \in \mathbb{R}_+^2$; states of nature $\tilde{s}$, $\bar{s} \in [0,\pi/2]$ with $\theta(\bar{x}) \neq \bar{s}$ and $\tilde{s} = \theta(\tilde{x})$, and both $\tilde{s}$, $\bar{s}$ irrational:



That the output correspondence defined above satisfies Axioms P1, P2, P4.1, P4.2, P5 and P6 may be easily verified. As for Axiom P0, consider an arbitrary $x \in \mathbb{R}_+^2$ and a closed subset $F \subset \mathbb{R}_+^2$. If $\langle 0,x \rangle \cap F$ is not empty, then $P_x^{-1}(F) = S$ by definition. If $\langle 0,x \rangle \cap F$ is empty, there are two cases to consider: - (i) if $\theta(x)$ is rational or if $F \cap \{y \mid \|y\| \leq \|x\|, \theta(y) \leq \theta(x)\}$ is empty, then $P_x^{-1}(F)$ is empty; (ii) if $\theta(x)$ is irrational and $F \cap \{y \mid \|y\| \leq \|x\|, \theta(y) \leq \theta(x)\}$ is not empty, then $P_x^{-1}(F) = \{\theta(x)\}$, a singleton in S. In any case, $P_x^{-1}(F)$ belongs to the Borel σ-algebra $\mathcal{B}$, verifying P0.

The input correspondence $L : \mathbb{R}_+^2 \times [0,\pi/2] \to 2^{(\mathbb{R}_+^2)}$ inversely related to P is given by

$$L(u,s) = \begin{cases} \{\lambda \cdot u \mid \lambda \geq 1\}, & \text{if } \left\{\begin{array}{l} s \text{ is rational; or} \\ \theta(u) = \pi/2 \text{ ; or} \\ s \text{ is irrational and } s \leq \theta(u) \end{array}\right\} ; \\ \{\lambda \cdot u \mid \lambda \geq 1\} \cup \{\beta \cdot v \mid \beta \geq 1, \|v\| = \|u\|, \theta(v) = s\}, & \text{if otherwise} . \end{cases}$$

Axioms L1, L2, L3, L4.1, L4.2, L5 and L6 may be easily verified for the correspondence L . As for L0, fix an arbitrary output u with $u > 0$ (i.e., $\theta(u) \in (0,\pi/2)$) . Consider the following closed set in the input space $X \equiv \mathbb{R}_+^2$:

$$F := \{x \in \mathbb{R}_+^2 \mid \|x\| = \|u\| \; , \; \theta(x) \in [\theta(u) + \varepsilon \; , \; \pi/2]\}$$

where $\varepsilon > 0$ and $\theta(u) + \varepsilon < \pi/2$. Observe that the set

$$\begin{aligned} L_u^{-1}(F) &\equiv \{s \in S \mid L(u,s) \cap F \neq \emptyset\} \\ &= \{s \in [0,\pi/2] \mid s \in [\theta(u) + \varepsilon \; , \; \pi/2] \text{ and } s \text{ irrational}\} \end{aligned}$$

which is clearly not an element of the Borel σ-algebra of $[0,\pi/2]$. Hence L0 does not hold $\square$

Since it is desirable to have the input correspondences satisfy L0, from now on, L0 *will be taken as an axiom on production technologies,* even though L0 is not implied by P0. As in Definition (1.4.6), the system {L0, L1, L2, L3, L4.1, L4.2, L5 & L6} will be referred to as the *stochastic weak axioms* on the stochastic input correspondences.

In the remainder of this section, some sufficient conditions for the validity of L0 are given. First, a useful measurability property on correspondences is stated (Notation: $\mathcal{B}(M)$ denotes the Borel σ-algebra of a metric space M ; $\otimes$ denotes the operation of forming product σ-algebra):

<u>(1.5.5) Proposition</u>: (Hildenbrand [1972, D.II.3-4]): Let H be a correspondence from a complete measure space $(S,\mathcal{S})$ to a complete separable metric space M .

(a) If the graph of H belongs to $\mathcal{S} \otimes \mathcal{B}(M)$, then for every $B \in \mathcal{B}(M)$, $\{s \in S \mid H(s) \cap B \neq \emptyset\} \in \mathcal{S}$.

(b) If for every open subset B of M, the set $\{s \in S \mid H(s) \cap B \neq \emptyset\}$ belongs to $\mathcal{S}$, then the graph of the closed-valued correspondence $\bar{H}$ defined by: $s \in S \mapsto \overline{H(s)}$; belongs to $\mathcal{S} \otimes \mathcal{B}(M)$.

(c) Statement (b) is valid if "open" is replaced by "closed".

(1.5.6) Proposition: Suppose Axiom P3 is replaced by the stronger axiom of free disposal of inputs P3.S in the system of stochastic weak axioms (1.4.6). Then a technology with a complete separable metric space X as its input space satisfies LO.

Proof: Consider an arbitrary open set G in the input space X and an arbitrary output $u \in U$. Let Z be a countable dense subset of G. Such a set Z exists since X is separable. It is obvious that

$$\bigcup_{z \in Z} \{s \in S \mid z \in L(u,s)\} \subset \{s \in S \mid L(u,s) \cap G \neq \emptyset\} .$$

To show the converse inclusion, consider an arbitrary state $s \in S$ for which there exists an input y belonging to $L(u,s) \cap G$. Since G is open, there is an open ball B centered at y with $B \subset G$. The free disposal of inputs (P3.S) implies $\{w \in X \mid w \geqq y\} \subset L(u,s)$. Clearly, the intersection set $\{w \in X \mid w \geqq y\} \cap B$ has a non-empty interior. Consequently, by the denseness of Z in G, there is a $z \in Z \cap B$ with $z \in L(u,s)$; establishing the converse inclusion.

Axiom PO ensures that for the arbitrarily chosen u, and every $z \in Z$, the set $\{s \in S \mid z \in L(u,s)\} = \{s \in S \mid \{u\} \cap P(z,s)\}$ is an element of $\mathcal{S}$. Hence, the set $\{s \in S \mid L(u,s) \cap G \neq \emptyset\}$, being a countable union of events, also belongs to $\mathcal{S}$.

Now Axiom P5 states that the correspondence $L_u : S \to 2(X)$ is closed-valued. By applying Proposition (1.5.5.c), then (1.5.5.a), it is seen that L_u is measurable. The proof is complete since u was arbitrarily chosen □

The simplest examples of complete separable metric spaces are $\mathbb{R}_+$ and $(l_1)_+$. Another example is $(L_1(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\mu))_+$, the non-negative orthant of the space of Borel integrable functions. Other examples include the (finite) product of the above spaces.

Since P3.S is not valid for many production systems, it is desirable to have sufficient conditions for LO without assuming P3.S. For this purpose, the following is useful:

<u>(1.5.7) Proposition</u>: Suppose X and U are both complete separable metric spaces. If an output correspondence $P : X \times S \to 2(U)$ not only satisfies the weak axioms (1.4.6) but also has each correspondence P_s $(s \in S)$ continuous (i.e., both upper and lower-hemi-continuous) and compact-valued, then LO holds.

<u>Proof</u>: Let Z be a countable dense subset of X . For each $z \in Z$, denote by $B^k(z)$ the open ball $\{x \in X \mid \|z - x\| < 1/k\}$ centered at z with radius $1/k$, $k = 1,2, \ldots$. Let F be a non-empty closed subset of U . For $k = 1,2, \ldots,$ let an open set $D^k(F)$ be defined by $D^k(F) := \{u \in U \mid d(u,F) < 1/k\}$ where $d(u,F)$ is the distance of the point u from the closed set F .

The continuity of the correspondences P_s $(s \in S)$ implies: for the output correspondence P , the inverse of F is given as:

$$P^{-1}(F) := \{(x,s) \in X \times S \mid P(x,s) \cap F \neq \emptyset\}$$
$$= \bigcap_{k=1}^{\infty} \bigcup_{z \in Z} [B^k(z) \times \{s \in S \mid P(z,s) \cap D^k(F) \neq \emptyset\}] .$$

First, it will be shown that the first set above is contained in the second. Let $(x^o,s^o) \in \{(x,s) \mid P(x,s) \cap F \neq \emptyset\}$. Then by the definition of $D^k(F)$, $P(x^o,s^o) \cap D^k(F)$ is not empty for each index k . Since $D^k(F)$ is open, and P_{s^o} is l.h.c., there exists for each k a neighborhood N_k of x^o such that $w \in N_k$ implies $P(w,s^o) \cap D^k(F) \neq \emptyset$. Since the set Z is dense in X , there exists for each k an input z^k with $z^k \in Z \cap N_k$ and $x^o \in B^k(z^k)$. For this z^k , clearly $(x^o,s^o) \in B^k(z^k) \times \{s \in S \mid P(z^k,s) \cap D^k(F) \neq \emptyset\}$. Hence (x^o,s^o) belongs to the second set.

To show the converse inclusion, let (x^o,s^o) belongs to the second set. Then there is an infinite sequence $\{z^k\} \subset Z$ with $(x^o,s^o) \in B^k(z^k) \times \{s \in S \mid P(z^k,s) \cap D^k(F) \neq \emptyset\}$, $k = 1,2, \ldots$. Clearly, $\{z^k\}$ converges to x^o . Let $\{v^k\}$ be an infinite sequence with $v^k \in P(z^k,s^o) \cap D^k(F)$. Since the correspondence P_{s^o} is compact-valued and u.h.c., there is a subsequence $\{v^j\} \subset \{v^k\}$ converging to a limit $v^o \in P(x^o,s^o)$; (see Hildenbrand [1972, B.III, Theorem 1]). Recall that $d(v^j,F) < 1/j$ for each j ; and F is closed. Thus $v^j \to v^o$ implies $v^o \in F$. Consequently, $P(x^o,s^o) \cap F \neq \emptyset$, establishing the converse inclusion.

Using Axioms P0, P5 and Proposition (1.5.5.c), one obtains that for each $z \in Z$ the graph of the correspondence P_z belongs to $\mathcal{S} \otimes \mathcal{B}(U)$. Then it follows from Proposition (1.5.5.a) that $\{s \in S \mid P(z,s) \cap D^k(F) \neq \emptyset\} \in \mathcal{S}$ for all $z \in Z$. Hence the set $[B^k(z) \times \{s \in S \mid P(z,s) \cap D^k(F) \neq \emptyset\}]$ belongs to $\mathcal{B}(X) \otimes \mathcal{S}$ for all $z \in Z$. Thus $P^{-1}(F)$ belongs

to $\mathcal{B}(X) \otimes \mathcal{S}$ since it may be obtained by countable union/intersection of elements of $\mathcal{B}(X) \otimes \mathcal{S}$. This conclusion is trivially true if $F = \emptyset$.

Since the correspondence $P : X \times S \to 2(U)$ is closed-valued (Axiom P5), and it was shown that $P^{-1}(F) \in \mathcal{B}(X) \otimes \mathcal{S}$ for all closed subset F of U, Proposition (1.5.5.c) applies to show that the graph of P (which is the technical feasibility set $\mathcal{T}$) belongs to $(\mathcal{B}(X) \otimes \mathcal{S}) \otimes \mathcal{B}(U)$.

Now let Y be an arbitrary closed subset of X, and u an arbitrary output in U. It follows from above that $(Y \times S \times \{u\}) \cap \mathcal{T}$ belongs to $\mathcal{B}(X) \otimes \mathcal{S} \otimes \mathcal{B}(U)$. For the input correspondence $L_u : S \to 2(X)$, the inverse image of Y is

$$\begin{aligned} L_u^{-1}(Y) &:= \{s \in S \mid L(u,s) \cap Y \neq \emptyset\} \\ &= \operatorname{Proj}_S \{(Y \times S \times \{u\}) \cap \mathcal{T}\} . \end{aligned}$$

Since $(S,\mathcal{S})$ is assumed to be a complete measure space, and $X \times U$ is a complete separable metric space, then the projection theorem (see Hildenbrand [1972, D.I.11]) applies to show that $L_u^{-1}(Y) \in \mathcal{S}$. Since Y and u are arbitrary, LO is established $\square$

Up until now, the only structure imposed on $(S,\mathcal{S})$ is that it is a complete measure space. If the state space S is a metric space, then the following assumption on a technology is meaningful:

P5.S The graph of the output correspondence $P : X \times S \to 2(U)$ is closed; i.e., the technical feasibility set $\mathcal{T}$ is closed.

<u>(1.5.8) Proposition</u>: Suppose S is a complete metric space. Furthermore, suppose P5.S above holds for a compact-valued output correspondence $P : X \times S \to 2(U)$. If the Borel σ-algebra $\mathcal{B}(S)$ is a sub-σ-algebra

of $\mathcal{S}$ and both X and U are complete separable metric spaces, then LO holds.

Proof: First it will be shown that P is measurable with respect to the σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(S)$. Suppose not so, then there exists a closed subset F of U whose inverse under P , namely the set $P^{-1}(F) := \{(x,s) \in X \times S \mid P(x,s) \cap F \neq \emptyset\}$, is not an element of $\mathcal{B}(X) \otimes \mathcal{B}(S)$. In particular, it is not a closed subset of $X \times S$. Hence, there is convergent sequence $\{(x^k,s^k)\} \subset P^{-1}(F)$ with a limit $(x^o,s^o) \notin P^{-1}(F)$. However, $(x^o,s^o) \in X \times S$ since both X and S are complete. Since $P(x^o,s^o) \cap F = \emptyset$, F is closed and $P(x^o,s^o)$ is compact, there is an open set G with $P(x^o,s^o) \subset G$ but $G \cap F = \emptyset$. Now since $P(x^k,s^k) \cap F \neq \emptyset$ for each k , $P(x^k,s^k) \not\subset G$. Hence P cannot be u.h.c. at (x^o,s^o) ; contradicting the hypothesis of the proposition.

With the measurability of P with respect to $\mathcal{B}(X) \otimes \mathcal{B}(S)$, an argument identical to the last part of the proof of Proposition (1.5.7) may be used to complete the proof since $\mathcal{B}(S) \subset \mathcal{S}$ □

1.6 Technical Efficiency

Given a stochastic technology $\mathcal{T}$, an input-output pair $(x,u) \in X \times U$ is called a *technical feasible production program* under state s if $(x,s,u) \in \mathcal{T}$. To evaluate the efficiency of the production programs, the following notion is useful:

(1.6.1) Definition: The collection of technically efficient inputs which may yield an output u under state s is called an *input efficient subset* and is defined by: for all $u \in U$, $s \in S$,

$$E(u,s) := \{x \in L(u,s) \mid y \leq x \text{ implies } y \notin L(u,s)\} .$$

To ensure that technical efficiency is not a vacuous concept, it is important to ascertain whether the efficient subsets as defined are not empty.

(1.6.2) Proposition (Shephard [1970]): If the input space of a technology is finite dimensional, then $E(u,s)$ is not empty whenever $L(u,s)$ is not empty.

However, if the input space is not finite dimensional, the input efficient subset may very well be empty, as demonstrated by the following:

(1.6.3) Example: Let functions $f^k : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by:

$$f^k(t) := \begin{cases} 1 & , t \in [0,k) ; \\ 1 + 1/k & , t \in [k,+\infty) . \end{cases} \qquad (k = 1,2, \ldots)$$

Clearly, the functions f^k's are elements in $(L_\infty)_+$, and they are decreasing; i.e., $f^k \geq f^{k+1}$ for all k.

Suppose a state space S is given as $S \equiv [1,2]$ with the Borel measure on the interval. With the input and output spaces $X = U \equiv (L_\infty)_+$, define an input correspondence $L : U \times S \to 2(X)$ by

$$L(u,s) := \begin{cases} \left\{ x \in (L_\infty)_+ \;\middle|\; \begin{array}{l} x = \lambda \cdot f^k \text{ where } \lambda \geq s \cdot \|u\| \\ \text{and } k \in \{1,2, \ldots\} \end{array} \right\} , \text{ if } u \neq 0 ; \\ (L_\infty)_+ , \text{ if } u = 0 . \end{cases}$$

That the correspondence L satisfies the stochastic weak axioms {L0, L1, L2, L3, L4.1, L4.2, L5 and L6} may be easily verified. Consider an

arbitrary output u with $\|u\| > 0$. Clearly $L(u,s) \neq \emptyset$ for all $s \in S$. For an arbitrary state $\bar{s} \in S$ and an arbitrary input $\bar{x} \in L(u,\bar{s})$, $\bar{x} = \bar{\lambda} \cdot f^p$ for some $\bar{\lambda} \geqq \bar{s} \cdot \|u\|$ and some integer p . By the definition of L , the inputs $\bar{\lambda} \cdot f^q \in L(u,\bar{s})$ for all $q \geq p$. Since the functions f^k's are decreasing, $\bar{x}$ cannot be an efficient input. Since $\bar{x}$, u and $\bar{s}$ were arbitrarily chosen $E(u,s)$ is empty for all $s \in S$ and $u \neq 0$ □

If the input space X is infinite dimensional, a weaker topology (than the norm topology) on X may ensure the non-emptiness of $E(u,s)$. For instance:

<u>(1.6.4) Proposition</u>: Suppose a stochastic technology satisfies L5 (equivalently P5) as stated with the weak* topology on $X \equiv (L_\infty)^n_+$. The efficient subset $E(u,s)$ is not empty if $L(u,s)$ is not empty.

<u>Proof</u>: Suppose $x \in L(u,s)$. Define the set

$$D_x := L(u,s) \cap \{y \in (L_\infty)^n_+ \mid y \leqq x\} .$$

Note that since the $\{y \in (L_\infty)^n_+ \mid y \leqq x\}$ is weak* closed, and $L(u,s)$ is also weak* closed (by L.5), D_x is weak* closed.

For each element w in D_x , define analogously $D_w := L(u,s) \cap \{y \in (L_\infty)^n_+ \mid y \leqq w\}$. Denote the collection of all such sets by $\mathfrak{M}$. Partially order the sets D_w's in $\mathfrak{M}$ by set inclusion. By the Hausdorff Maximality Theorem, there is a maximal linearly ordered sub-collection $\mathfrak{M}'$ of $\mathfrak{M}$. Obviously, the sub-collection $\mathfrak{M}'$ has the finite intersection property. Note that each D_w in $\mathfrak{M}'$ is weak* closed. Furthermore, by the theorem of Alaoglu, they are actually weak*

compact. Consequently, the intersection set $\cap \{D_w \mid D_w \in \mathcal{H}'\}$ is not empty. Let w^* be an element of this intersection set. Clearly $w^* \in L(u,s)$. Suppose $w^* \notin E(u,s)$. Then there exists an input $z \in L(u,s)$ with $z \leq w^*$. Then the set $D_z := L(u,s) \cap \{y \mid y \leq z\}$ is non-empty and is a proper subset of every element of $\mathcal{H}'$, contradicting the maximality of $\mathcal{H}'$. Hence $w^* \in E(u,s)$ □

The general condition under which the technical efficient subset being non-empty is not known. However, in almost all models of production, the non-emptiness of the efficient subset is assumed. This practice will be followed in the subsequent exposition.

The following straight forward fact will be useful later:

<u>(1.6.5) Fact</u>: Suppose $E(u,s) \neq \emptyset$, then $L(u,s) \subset E(u,s) + X$.

Intuitively speaking, technical efficiency cannot prevail when inputs of infinitely large size are used to yield a finite output. This notion is formalized by the following *asymmetric axiom* on the input correspondences.

E For each state $s \in S$ and $u \in U$, $E(u,s)$ is bounded.

For a detailed discussion of the significance of this axiom, see Shephard [1970-a]. Here, it is remarked if $X \equiv (L_\infty)^n_+$ and L5 is stated with the weak* topology on $(L_\infty)^n_+$, then Axiom E implies that the weak* closure of $E(u,s)$ is weak* compact. This fact will be useful in the next Chapter.

1.7 Confidence Indexed Production Correspondences

In applications, one may not be interested in the total structure of the stochastic production correspondence P. Rather, one could be primarily concerned with the issues:

(1.7.1) With what probability a certain subset of outputs may be obtained from a specified subset of inputs?

(1.7.2) Given that a certain level of outputs is to be attained with at least a probability ξ, what inputs are feasible?

To address these issues, it is convenient to define:

(1.7.3) Definition: For a stochastic technology $\mathcal{T}$ represented by an output correspondence $P : X \times S \to 2(U)$, its associated *confidence indexed output correspondence* CP is given by:

$$(x,\xi) \in X \times [0,1] \longmapsto CP(x,\xi) := \{u \in U \mid \mathcal{P}\{s \in S \mid u \in P(x,s)\} \geqq \xi\} .$$

The word "confidence" need not pertain to the subjective belief of any producer; it is used simply to denote a probability measure of certain events.

For each scalar $\xi \in [0,1]$, let $\mathcal{A}(\xi)$ be the collection of events $\{A \in \mathcal{B} \mid P(A) \geqq \xi\}$. Then the confidence indexed output sets $CP(x,\xi)$ may be equivalently defined as:

$$CP(x,\xi) \equiv \bigcup_{A \in \mathcal{A}(\xi)} \bigcap_{s \in A} P(x,s) , \text{ all } x \in X , \xi \in [0,1] .$$

The following properties for the correspondence CP are suggested by the axioms on P :

CP1 $0 \in CP(x,\xi)$ for all $x \in X$ and $\xi \in [0,1]$; $CP(x = 0 , \xi) \equiv \{0\}$ if $\xi > 0$; and $CP(x , \xi = 0) \equiv U$ for all $x \in X$.

CP2 For all $x \in X$, $CP(x,\xi)$ is bounded if $\xi > 0$.

CP3 For all $x \in X$ and $\xi \in [0,1]$, $CP(x,\xi) \subset CP(\lambda \cdot x,\xi)$ if $\lambda \geqq 1$.

CP3.S For all $\xi \in [0,1]$, $CP(x,\xi) \subset CP(y,\xi)$ if $y \geqq x$.

CP4.1 For each $i \in \{1, \ldots, m\}$, there exists an output u with $u_i \neq 0$, an input x and a confidence level $\xi > 0$ such that $u \in CP(x,\xi)$.

CP4.2 If an output $u \neq 0$ and $u \in CP(x,\xi)$, then for every scalar $\theta > 0$ there exists a positive scalar $\lambda_{\theta,\xi}$ such that $\theta \cdot u \in CP(\lambda_{\theta,\xi} \cdot x,\xi)$.

CP5 The graph of the correspondence CP is closed; i.e., $x^k \to x^o$, $u^k \to u^o$, $\xi^k \to \xi^o$ and $u^k \in CP(x^k,\xi^k)$ for each k implies $u^o \in P(x^o,\xi^o)$.

CP6 For every $\xi \in [0,1]$ and $x \in X$, $u \in CP(x,\xi)$ and $\theta \in [0,1]$ implies $\theta \cdot u \in P(x,\xi)$.

CP6.S For every $\xi \in [0,1]$ and $x \in X$, $u \in CP(x,\xi)$ and $v \leqq u$ implies $v \in CP(x,\xi)$.

CP7 For every $x \in X$, $\xi \geqq \xi'$ implies $CP(x,\xi) \subset CP(x,\xi')$.

Note that other than having the confidence index ξ , properties CP1 to CP6 are almost identical to the axioms on a deterministic technology formulated in Shephard [1974] and Shephard/Färe [1980]. In this sense, the confidence index ξ may be regarded as an "input" to production,

with the peculiar monotonicity property CP7.

<u>(1.7.4) Proposition</u>: The properties CP1, CP2, CP3 (CP3.S), CP4.1, CP6 (CP6.S) and CP7 may be derived from the axioms P0, P1, P2, P3 (P3.S), P4.1, P4.2 and P6 (P6.S).

<u>Proof</u>: (CP1) Since $0 \in P(x,s)$ for every $x \in X$ and $s \in S$, $0 \in CP(x,\xi)$ for every $x \in X$ and $\xi \in [0,1]$. Moreover, since $P(x = 0, s) \equiv \{0\}$ for each state $s \in S$, $CP(x = 0, \xi) = \{0\}$ for all $\xi > 0$. By P0 and the assumption that $(S, \mathcal{S}, \mathcal{P})$ is complete, for each fixed input $x \in X$ and output $u \in U$, the set $\{s \in S \mid u \in P(x,s)\}$ is an event and has at least zero probability. Hence $CP(x, \xi = 0) = U$ for each $x \in X$, including the case of $x \equiv 0$.

(CP2) Consider an arbitrary fixed input $x \in X$. For each positive scalar K define a set $A(K) := \{s \in S \mid P(x,s) \cap \{u \in U \mid \|u\| \geq K\} \neq \emptyset\}$. Since the set $\{u \in U \mid \|u\| \geq K\}$ is closed, by P0, $A(K) \in \mathcal{S}$ for all $K \in \mathbb{R}_{++}$. It is to be shown that for each $\alpha \in (0,1)$, there exists a positive scalar K_α (depending also on x) with $\mathcal{P}(A(K_\alpha)) < \alpha$. Suppose otherwise, then there is an increasing sequence $\{K^j\} \subset \mathbb{R}_{++}$ diverging to $+\infty$ but $\mathcal{P}(A(K^j)) \geq \alpha$ for each index j. Since $\{K^j\}$ is increasing, the sequence of events $\{A(K^j)\}$ is non-increasing. By the sequential continuity of probability measures, the set $A := \bigcap_j A(K^j)$ is an event and has probability $\mathcal{P}(A) \geq \alpha$. Clearly, $P(x,s)$ is not bounded for each $s \in A$; contradicting Axiom P2. Hence, for every confidence level $\xi \in (0,1)$, $CP(x,\xi)$ is bounded (in norm) by the positive scalar $K_{(1-\xi)}$. Finally, by property CP7 (to be shown later), $CP(x, \xi = 1) \subset CP(x,\xi')$ if $\xi' < 1$. Hence $CP(x,1)$ is also bounded. Since the input x was arbitrarily chosen, CP2 is established.

(CP3, CP3.S, CP4.1, CP6, CP6.S) If $u \in CP(x,\xi)$, then there is an event A with $\mathcal{P}(A) \geqq \xi$ such that $u \in P(x,s)$ for each $s \in A$. Axiom P3 states that for all $\lambda \geqq 1$, $u \in P(\lambda \cdot x,s)$ for each $s \in A$. Hence, $u \in CP(\lambda \cdot x,\xi)$; establishing CP3. Analogously, CP3.S follows from P3.S. Similar arguments may be used to establish CP6 and CP6.S from P6 and P6.S. Property CP4.1 is merely a restatement of P4.1.

(CP7) Recall the definition of the collection $\mathcal{B}(\xi)$ of events. For $\xi \geqq \xi'$, it is clear that $\mathcal{B}(\xi) \subset \mathcal{B}(\xi')$. Thus, for every $x \in X$,

$$CP(x,\xi) = \bigcup_{A \in \mathcal{B}(\xi)} \bigcap_{s \in A} P(x,s) \subset \bigcup_{A \in \mathcal{B}(\xi')} \bigcap_{s \in A} P(x,s) = CP(x,\xi') \ \square$$

To ensure the property CP5 and CP4.2 to be valid, stronger conditions than those of the stochastic weak axioms on the technology seem to be necessary. A derivation of CP5 is given in the next proposition.

<u>(1.7.5) Proposition</u>: Suppose the output correspondence $P : X \times S \to 2^{(U)}$ is upper-hemi-continuous, and the input space X is a separable metric space. Then the confidence indexed correspondence CP associated with P has property CP5.

<u>Proof</u>: Suppose infinite sequences $\{x^k\} \subset X$, $\{u^k\} \subset X$ and $\{\xi^k\} \subset [0,1]$ have $x^k \to x^o$, $u^k \to u^o$, $\xi^k \to \xi^o$ and $u^k \in CP(x^k,\xi^k)$ for all k ; it is to be shown that $u^o \in CP(x^o,\xi^o)$. Since $CP(x , \xi = 0) \equiv U$ (property CP1) for all $x \in X$, one may assume without loss of generality that $\xi^o > 0$.

Let d be the norm metric on X . Define a function f by:

$$s \in S \to f(s) := \text{Inf } \{d(u^o,v) \mid v \in P(x^o,s)\} .$$

Note that for all $\alpha \in \mathbb{R}_{++}$, $\{s \in S \mid f(s) < \alpha\} = \{s \in S \mid \bar{B}(u^o,\alpha) \cap$

$P(x^o,s) \neq \emptyset\}$ where $\bar{B}(u^o,\alpha)$ is the closed ball centered at u^o with radius α. By Axiom PO as applied on the correspondence P_{x^o}, $\{s \in S \mid f(s) < \alpha\} \in \mathcal{B}$ for all $\alpha \in \mathbb{R}_{++}$. This, together with the fact that f is non-negative-valued, implies that f is a measurable function on S.

Next, define a sequence of events:

$$A^k := \{s \in S \mid u^k \in P(x^k,s)\} \ , \ k = o,1,2, \ldots \ .$$

Consider the restriction of f to the complement of A^o; i.e., $(A^o)^c := S\backslash A^o$, the following claim is made: For each $\varepsilon \in (0,1]$ there exists a $\delta \in \mathbb{R}_{++}$ such that the event $D(\delta) := \{s \in (A^o)^c \mid f(s) \leqq \delta\}$ has probability $\mathcal{P}(D(\delta)) \leqq \varepsilon$.

To show the claim, suppose it is false. Then there is a monotone sequence of scalars $\{\delta^j\} \subset \mathbb{R}_{++}$ decreasing to 0 such that $\mathcal{P}(D(\delta^j)) > \varepsilon$ for each index j. By definition, $D(\delta^{j+1}) \subset D(\delta^j)$ for all j. Then by the sequential continuity of probability measures, the set $D := \bigcap_j D(\delta^j)$ is an event and $\mathcal{P}(D) \geqq \varepsilon > 0$. It follows from the definition of f that for each $s \in D$, there is no open ball $B(u^o)$ centered at u^o with $B(u^o) \subset U\backslash P(x^o,s)$. This contradicts the fact that $D \subset (A^o)^c$ (i.e., $u^o \notin P(x^o,s)$ for all $s \in D$) and $P(x^o,s)$ is closed for all $s \in S$.

Now let an $\varepsilon \in (0,1]$ be arbitrarily fixed together with an associated δ and event $D(\delta)$. Let $B(u^o,\delta/4)$ be the open ball centered at u^o with radius $\delta/4$. By construction, $B(u^o,\delta/4) \cap P(x^o,s) = \emptyset$ for each $s \in (A^o)^c\backslash D(\delta)$. Let G be the open set defined by $G := U\backslash \bar{B}(u^o,\delta/2)$ where $\bar{B}(u^o,\delta/2)$ is the closed ball at u^o with radius $\delta/2$. Since for all $s \in (A^o)^c\backslash D(\delta)$, every element of

$B(u^o,\delta/4)$ is at least of distance $3\delta/4$ from $P(x^o,s)$, it follows that $P(x^o,s) \subset G$ on $(A^o)^c \setminus D(\delta)$.

The following fact (to be proved as the next proposition) results from the u.h.c. of the correspondences P_s at x^o: there exists for each $\beta \in (0,1]$ an event H_β contained in $(A^o)^c \setminus D(\delta)$ with $\mathcal{P}(H_\beta) \leqq \beta$ such that for some neighborhood N of x^o, $P(w,s) \subset G$ for each $w \in N$ and $s \in (A^o)^c \setminus D(\delta) \setminus H_\beta$.

Without loss of generality, let $\beta = \varepsilon$. Furthermore, let H_ε and N be an event and neighborhood for which the conclusion of the above fact holds. Since $x^k \to x^o$ and $u^k \to u^o$, there is an integer K such that for all $k \geqq K$, $u^k \in B(u^o,\delta/4)$ and $x^k \in N$. Then it follows that for all $k \geqq K$ and all $s \in (A^o)^c \setminus D(\delta) \setminus H_\varepsilon$, $P(x^k,s) \subset G$; consequently $u^k \notin P(x^k,s)$ since $B(u^o,\delta/4) \subset G^c$ by construction. Recalling the definition of the events A^k, one then has: for all $k \geqq K$, $(A^o)^c \setminus D(\delta) \setminus H_\varepsilon \subset (A^k)^c$. In other words, $A^k \subset A^o \cup D(\delta) \cup H_\varepsilon$ since $D(\delta) \subset (A^o)^c$ and $H_\varepsilon \subset (A^o)^c \setminus D(\delta)$. Hence $\xi^k \leqq \mathcal{P}(A^k) \leqq \mathcal{P}(A^o) + \varepsilon + \varepsilon$ for all $k \geqq K$. Since $\xi^k \to \xi^o$, at the limit, $\xi^o \leqq \mathcal{P}(A^o) + 2\varepsilon$. By letting ε become arbitrarily small, it is seen that $\mathcal{P}(A^o) \geqq \xi^o$; i.e., $u^o \in CP(x^o,\xi^o)$ $\square$

The unproved fact invoked in the above proof is of independent interest, and is stated here as:

<u>(1.7.6) Proposition</u>: Let $(S,\mathcal{S},\mathcal{P})$ be a probability space, X a separable metric space and U a topological space. Suppose a correspondence $P : X \times S \to 2(U)$ satisfies: (a) for each $s \in S$, the correspondence $x \in X \to P_s(x) := P(x,s)$ is u.h.c.; (b) for each $x \in X$, the correspondence $s \in S \to P_x(s) := P(x,s)$ is measurable.

Consider arbitrarily an $\bar{x} \in X$, an open subset G of U and an event $A \in \mathcal{B}$. If $P(\bar{x},s) \subset G$ for all $s \in A$, then for every $\beta \in (0,1]$ there exists an event $H \subset A$ with $\Phi(H) \leq \beta$ and a neighborhood N of $\bar{x}$ such that for all $w \in N$ and $s \in A \backslash H$, $P(w,s) \subset G$.

<u>Proof</u>: Let $\bar{x} \in X$, open $G \subset U$ and $A \in \mathcal{B}$ satisfy $P(\bar{x},s) \subset G$ for all $s \in A$. Define (while focusing attention to the restriction of $\mathcal{B}$ on A) a function $g : A \to \mathbb{R}_{++}$

$$s \in A \longmapsto g(s) := \operatorname{Sup} \{r \in \mathbb{R}_{++} \mid P(y,s) \subset G \quad \text{for all} \quad y \in B(\bar{x},r)\}$$

where $B(\bar{x},r)$ is the open ball centered at $\bar{x}$ with radius r. The positive-valued function g is well-defined since the correspondences P_s are u.h.c. on A. To show that g is measurable on A, first note that

$$(1.7.6.1) \quad \{s \in A \mid g(s) \geq r\} = \{s \in A \mid P(y,s) \subset G \quad \text{for all} \quad y \in B(\bar{x},r)\}$$

for all $r \in \mathbb{R}_{++}$. Fix an arbitrary $\bar{r} \in \mathbb{R}_{++}$ and let Z be a countable dense subset of $B(\bar{x},\bar{r})$. Clearly,

$$\{s \in A \mid P(y,s) \subset G \quad \text{for all} \quad y \in B(\bar{x},\bar{r})\} \subset \bigcap_{z \in Z} \{s \in A \mid P(z,s) \subset G\} .$$

To show the converse inclusion, let $\bar{s}$ belong to the intersection set on the right. Since the correspondence $P_{\bar{s}}$ is u.h.c., there exists for each $z \in Z$ an neighborhood N_z of z such that $P(w,\bar{s}) \subset G$ for all $w \in N_z$. Since Z is dense in $B(\bar{x},\bar{r})$, it may be verified that the collection $\{N_z \, ; \, z \in Z\}$ is an open covering of $B(\bar{x},\bar{r})$. Hence $P(y,\bar{s}) \subset G$ for all $y \in B(\bar{x},\bar{r})$; establishing the converse inclusion. Now, for each $z \in Z$, the set $\{s \in A \mid P(z,s) \subset G\}$ belongs to $\mathcal{B}$

since it is equal to $A \setminus \{s \in A \mid P(z,s) \cap G^c \neq \emptyset\}$ which is an element of $\mathscr{B}$ according to assumption (b) . Hence, the set $\{s \in S \mid g(s) \geq \bar{r}\} \in \mathscr{B}$ since it is a countable intersection of elements in $\mathscr{B}$. Since $\bar{r}$ was arbitrarily chosen and g is positive-valued, g is measurable on A .

A contra-positive argument is now used to complete the proof. Suppose the conclusion of the proposition is not true for some $\beta \in (0,1]$. Without loss of generality, assume $\mathcal{P}(A) > \beta > 0$. Then for every positive integer k and its corresponding open ball $B(\bar{x},1/k)$,

$$\{s \in A \mid P(y,s) \subset G \text{ for all } y \in B(\bar{x},1/k)\} < \mathcal{P}(A) - \beta \text{ .}$$

Define $W^k := \{s \in A \mid g(s) \geq 1/k\}$, $k = 1,2, \ldots$. Since g is measurable, $W^k \in \mathscr{B}$; and recalling identity (1.7.6.1), the above inequality is equivalent to: $\mathcal{P}(W^k) < \mathcal{P}(A) - \beta$, $k = 1,2, \ldots$. Clearly $W^{k+1} \supset W^k$ for all k . Let $W := \bigcup_k W_k$. By the sequential continuity of probability, $\mathcal{P}(W) \leq \mathcal{P}(A) - \beta$. In other words, $P\{s \in A \mid g(s) > 0\} \equiv \mathcal{P}(W) \leq \mathcal{P}(A) - \beta < \mathcal{P}(A)$ since $\beta > 0$. This contradicts the fact that g is positive-valued on A □

As for the property CP4.2, the following shows that it is almost always valid:

<u>(1.7.7) Proposition</u>: CP4.2 holds for the confidence index $\bar{\xi} = 0$. For the case $\bar{\xi} > 0$, suppose $\bar{u} \neq 0$ and $\bar{u} \in CP(\bar{x},\bar{\xi})$. Then for every $\theta \in \mathbb{R}_{++}$ and every $\xi \in [0,\bar{\xi})$, there exists a $\lambda \in \mathbb{R}_{++}$ such that $\theta \cdot \bar{u} \in CP(\lambda \cdot \bar{x},\xi)$.

<u>Proof</u>: The first statement is trivially true since $CP(x,0) \equiv U$. Let

$\tilde{\xi} > 0$ and $\bar{u} \in CP(\bar{x},\tilde{\xi})$. Suppose $\bar{\theta} \in \mathbb{R}_{++}$ and $\bar{\xi} \in [0,\tilde{\xi})$ is such that for all $\lambda \in \mathbb{R}_{++}$, $\bar{\theta}\cdot\bar{u} \notin CP(\lambda\cdot\bar{x},\bar{\xi})$. Let an infinite sequence $\{\lambda^k\} \subset \mathbb{R}_{++}$ diverges $+\infty$. Define for each index k , $W^k := \{s \in S \mid P(\lambda^k\cdot x,s) \cap \{\sigma\cdot\bar{u} \mid \sigma \geqq \bar{\theta}\} \neq \emptyset\}$. By Axiom PO, $W^k \in \mathcal{S}$ for all k since the set $\{\sigma\cdot\bar{u} \mid \sigma \geqq \bar{\theta}\}$ is closed in U . By the contra-positive hypothesis, $\mathcal{P}(W^k) < \bar{\xi}$ for all k . Now clearly the sequence of events $\{W^k\}$ is monotone non-decreasing. Let $W := \bigcup_k W^k$. By the sequential continuity of probability measures, $\mathcal{P}(W) \leqq \bar{\xi} < \tilde{\xi}$. Hence the complement of W , namely W^c , is not empty. Note that W^c has a non-empty intersection with the event $\{s \in S \mid \bar{u} \in P(\bar{x},s)\}$. Consider an arbitrary $\bar{s} \in W^c$ with $\bar{u} \in P(\bar{x},\bar{s})$. The fact that $\bar{s} \in W^c$ implies there does not exist $\lambda \in \mathbb{R}_{++}$ with $\bar{\theta}\cdot\bar{u} \in P(\lambda\cdot\bar{x},\bar{s})$; contradicting Axiom P4.2 □

The following straightforward fact (proof omitted) shows that property CP4.2 for CP is guaranteed if one is willing to impose a considerably stronger attainability assumption on P than P4.2:

<u>(1.7.8) Fact</u>: Property CP4.2 holds if and only if for every $x \in X$, $u \in U$ and $\theta \in \mathbb{R}_{++}$, there exists an almost surely bounded function $\lambda : S \to \mathbb{R}_{++}$ such that $\theta\cdot u \in P(\lambda(s)\cdot x,s)$ on the event $\{s \in S \mid u \in P(x,s)\}$.

Although the properties CP4.2 and CP5 cannot be derived from the set of stochastic weak axioms on P , in view of (1.7.5) and (1.7.8), they nevertheless appear to be reasonable for the confidence indexed output correspondence of many real-world stochastic production technologies.

With the construction of the correspondence CP from P , the questions (1.7.1) and (1.7.2) posed at the beginning of this section may be answered. For ease of exposition, let the graph of the correspondence CP , i.e., $\{(x,\xi,u) \in X \times [0,1] \times \xi \mid u \in CP(x,\xi)\}$, be denoted by $\mathcal{GT}$.

The set $\mathcal{ST}$ may be regarded as the confidence indexed technical feasibility set.

Given a set of outputs $V \subset U$ and a set of inputs $Y \subset X$, consider the intersection set $D := (Y \times [0,1] \times V) \cap \mathcal{ST}$. Note that by property CP7, the set $\Xi(Y,V) := \{\xi \in [0,1] \mid \text{for some } y \in Y \text{ and } v \in V,\ (y,\xi,v) \in D\}$ is an interval containing 0. If both V and Y are closed sets, then D is closed; and thus so is $\Xi(Y,V)$ if CP5 holds. Then the least upper bound of $\Xi(Y,V)$; . ., the maximal probability with which some output in V may be obtained using inputs from the set Y, is actually attained. Thus, question (1.7.1) is resolved. Question (1.7.2) may be addressed in a similar manner.

It is interesting to relate the notion of a confidence indexed output correspondence to that of a stochastic production function. Suppose a technology has an input space $X \equiv \mathbb{R}_+^n$ and an output space $U \equiv \mathbb{R}_+$, and its confidence indexed output correspondence satisfies properties CP1 to CP7. Consider an arbitrary input $x \in \mathbb{R}_+^n$. Define a function F_x by

$$u \in \mathbb{R}_+ \longmapsto F_x(u) := \operatorname{Max}\{\xi \in [0,1] \mid u \in CP(x,\xi)\} .$$

The function F_x is well defined since according to CP5, the graph of CP is closed. Furthermore, it has the following properties as may be, easily verified:

(i) $F_x(0) = 1$;

(ii) $F_x(u) \geqq F_x(v)$ if $v \geqq u$;

(iii) $F_x(u) \downarrow 0$ as $u \to +\infty$;

(iv) F_x is upper-semi-continuous (u.s.c.).

Note that the function F_x resembles the distribution of a non-negative random variable. In fact, for a given input x, let Φ_x be a non-negative function on the state space S defined by:

$$(1.7.9) \qquad \Phi_x(s) := \operatorname{Max}\,\{u \in \mathbb{R}_+ \mid u \in P(x,s)\}\ ,\ s \in S\ .$$

Φ_x is well defined since $P(x,s)$ is compact (Axioms P2 and P5) and it is a bona-fide random variable according to Axiom P0. Φ_x has the survival distribution function

$$t \in \mathbb{R}_+ \longmapsto G(t) := P\{s \in S \mid \Phi_x(s) \geqq t\}\ .$$

It is easy to see that:

$$G(t) \geqq \xi \text{ if and only if } F_x(t) \geqq \xi\ ,\ t \in \mathbb{R}_+\ ,\ \xi \in [0,1]\ .$$

Hence, for each $x \in \mathbb{R}^n_+$, the function F_x gives the probability distribution of Φ_x, the random maximal output attainable using input x.

In view of the above discussion, the practice of specifying a family of random variables $\{\Phi_x\ ;\ x \in X \equiv \mathbb{R}^n_+\}$ as a model of production under uncertainty may be considered as only giving the confidence indexed correspondence CP associated with a technology; via the specification of the distribution functions F_x for all $x \in X$. (Note that for all $x \in X$ and all $\xi \in [0,1]$, $CP(x,\xi) \equiv \{u \in \mathbb{R}_+ \mid F_x(u) \geqq \xi\}$.) Since the correspondence CP is only a sketchy and aggregated representation of the underlying technology, the family $\{\Phi_x\ ;\ x \in X\}$ may not be a descriptive enough model to serve as a basis for making production decisions under uncertainty.

For example, a Cobb-Douglas type stochastic production model may be formulated as:

$$x \in \mathbb{R}^n_+ \to \phi_x := \beta \cdot \prod_{i=1}^{n} x_i^{\alpha_i} ,$$

where β and α_i's are non-negative dependent random variables, with $\Sigma\alpha_i \equiv 1$. The use of only a finite number of random parameters gives a compact representation of the family $\{\phi_x ; x \in X\}$, and may be convenient for econometric studies. However, it seems quite difficult to relate such a model to an explicit description of the stochastic production environment the characterization of which may be fundamental for making production decisions.

For completeness sake, the *confidence indexed input correspondence* CL inversely related to CP is defined. Formally, $CL : U \times [0,1] \to 2(X)$;

$$\begin{aligned} CL(u,\xi) &:= \{x \in X \mid \mathcal{P}\{s \in S \mid x \in L(u,s)\} \geqq \xi\} \\ &= \{x \in X \mid \mathcal{P}\{s \in S \mid u \in P(x,s)\} \geqq \xi\} \qquad (1.7.10) \\ &= \{x \in X \mid u \in CP(x,\xi)\} . \end{aligned}$$

The properties of the correspondence CL induced by those of CP (i.e., CP1 through CP7) are stated as follows:

CL1 $CL(u = 0 , \xi) = CL(u , \xi = 0) = X$; $0 \notin CL(u,\xi)$ if $u \neq 0$ and $\xi > 0$.

CL2 If $\xi > 0$ and $\|u^k\| \to +\infty$, then $\bigcap_k CL(u^k,\xi)$ is empty.

CL3 For all $u \in U$ and $\xi \in [0,1]$, if $x \in CL(u,\xi)$ and $\lambda \geqq 1$, then $\lambda \cdot x \in CL(u,\xi)$.

CL3.S For all $u \in U$ and $\xi \in [0,1]$, if $x \in CL(u,\xi)$ and $y \geqq x$, then $y \in CL(u,\xi)$.

CL4.1 For each $i \in \{1, \ldots, m\}$, there exists a scalar $\xi \in (0,1]$ and an output u with $u_i \neq 0$ such that $CL(u,\xi)$ is not empty.

CL4.2 If $x \in CL(u,\xi)$ and $u \neq 0$, then for every $\theta \in \mathbb{R}_{++}$, $\{\lambda \cdot x \mid \lambda \geqq 0\} \cap CL(\theta \cdot u,\xi)$ is not empty.

CL5 The graph of the correspondence CL is closed.

CL6 For all $u \in U$ and $\xi \in [0,1]$, $CL(u,\xi) \supset CL(\theta \cdot u,\xi)$ if $\theta \geqq 1$.

CL6.S For all $u \in U$ and $\xi \in [0,1]$, $CL(u,\xi) \supset CL(v,\xi)$ if $v \geqq u$.

CL7 For all $u \in U$, $CL(u,\xi) \supset CL(u,\xi')$ if $\xi' \geqq \xi$.

1.8 Information and Production Policies

In section 1 through 7, a model of stochastic production technology and various representations of it were given. This model only characterizes the purely technical aspects of production, and as such, is not a model of production decision making under uncertainty. This section introduces a notion of production policies which will be useful in later chapters.

In a deterministic model of production, e.g., the model of Shephard [1970-a], a notion of production policies is implicitly introduced when the so-called minimal-cost function is defined. For an output $u \in U \equiv \mathbb{R}^m_+$ and an input price vector $p \in \mathbb{R}^n_+$, the minimal cost of production is given as:

$$(u,p) \to Q(u,p) := \operatorname{Inf}\{p \cdot x \mid x \in L(u)\} ;$$

where $L(u)$ is the deterministic analogue of $L(u,s)$. Presumably, when faced with the market input price p, a producer chooses an input which yields the output u at a minimal cost. Thus, implicitly, it is assumed that every *selection* of input x from the set of technically feasible inputs $L(u)$ is a feasible input policy.

The notion of selection may be generalized for the case of a stochastic correspondence as follows:

(1.8.1) Definition: Let H be a correspondence from a measure space $(S,\mathcal{B},\mathcal{P})$ to a metric space M. A function $f : S \to M$ is a *selection* from H if $f(s) \in H(s)$ for all $s \in S$; an *almost everywhere selection* if $f(s) \in H(s)$ a.e., a *measurable selection* if f is measurable.

For convenience of exposition, in the remainder of this section, the input space X and the output space U are taken to be $(l_\infty)^n_+$ and $(l_\infty)^m_+$ respectively.

With the introduction of uncertainty, it is intuitively obvious that any reasonably well defined production policy must be concerned with any "information" on the unknown production environment. The model of information introduced in the following is that of Radner [1968]:

$\mathcal{V}$ is a partition of $(S,\mathcal{B})$ if $\mathcal{V} \equiv \{V_i\}$ is a collection of pairwise disjoint elements of the σ-algebra $\mathcal{B}$ with $\bigcup_i V_i = S$.

An *information structure* (of a producer) is an infinite-tuple $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_t, \ldots)$ where each $\mathcal{J}_t$ is a partition of the state

space $(S,\mathcal{S})$. See Radner [1968] for an interpretation of this abstract formulation of information.

For two partitions $\mathcal{W}$ and $\mathcal{V}$ of $(S,\mathcal{S})$, $\mathcal{W}$ is said to be *as fine as* (or as informative as) $\mathcal{V}$ if for every $W \in \mathcal{W}$ and $V \in \mathcal{V}$, either $W \subset V$ or $W \cap V = \emptyset$.

The "fineness" of the partition $\mathcal{J}_t$ indicates how informed a producer is at time t concerning the true state. Furthermore, the notion of "as fine as" provides a partial-ordering which will be used in Chapter 2 to define a notion of boundedness of information.

In the following, it is argued that the information structure of a producer, when imposed on the underlying stochastic technology (which is independent of the producer), determines the production policies (options) open to him. Abstractly, at time t, with the information $\mathcal{J}_t$ available, a producer engages in production by taking certain *actions* (procures inputs, allocates resources, commences certain production operations, etc.). Referring to the collection of possible actions as $\mathcal{A}$, the choice of action (production decision) over time is a mapping

$$\text{(1.8.2)}\qquad \begin{gathered} D : S \to \mathcal{A} \\ s \in S \to D(s) \equiv (D_1(s), D_2(s), \ldots, D_t(s), \ldots) \in \mathcal{A} \end{gathered}$$

where $D_t(s)$ denotes the action at time t. To be consistent with the information available, the following condition on the decision D holds:

$$\text{(1.8.3)}\qquad s', s'' \in I_t \in \mathcal{J}_t \Rightarrow D_t(s') = D_t(s'') ;$$

i.e., with an equivalent information as represented by I_t, the action taken at time t must be the same.

From a system theoretical point of view, the production decision of a producer is manifested by the associated inputs and outputs, which of course depend on the state of production environment. Explicitly, associated with decision D are the input and output mappings $\underline{x}$ and $\underline{u}$:

$$(D(s),s) \to (\underline{x}(s),\underline{u}(s)) \in (l_\infty)_+^n \times (l_\infty)_+^m$$

In order that production is technically feasible, the following condition on the mappings $\underline{x}$ and $\underline{u}$ must hold:

(1.8.4) $\qquad \underline{u}(s) \in P(\underline{x}(s),s) \qquad (s \in S\ ,\ D(s) \in \mathcal{A})$.

The *information consistency* and *technical feasibility* conditions, (1.8.3) and (1.8.4), limit the class of selections from the stochastic production correspondences P which may be meaningfully called production policies.

It was emphasized in the introduction that a model of technology should be free from any institutional constraints of the producer. The forgoing discussion indicates that to model the actual operation of a production unit under uncertainty, the *institutional* constraint of its information structure appears to be necessary. Moreover, in every existing production system, the means of production are always constrained (at least in the short-run); for instance, by its plant capacity, labor availability etc. Hence the above model of production policies is more appropriate as a planning model for as yet non-existing

production systems since the production correspondence has not been constrained. Suppose a production system is constrained by requiring its use of input of goods and services to be in a set $C \subset X$, then the technical feasibility constraint (1.8.4) is to be modified to

$$(1.8.4') \qquad \begin{array}{l} \underline{u}(s) \in P(\underline{x}(s),s) \\ \underline{x}(s) \in C \end{array} \qquad (s \in S \; , \; D(s) \in \mathcal{A}) \; .$$

Note that this modification indirectly imposes a restriction on the set of decisions open to a producer. Perhaps it should be remarked here that the space of decisions $\mathcal{A}$ in general is difficult to formalize. However, in some cases, they are quite explicit. For example, in Example (1.4.3), the choice of the intensity levels z^t, outputs u^t and the intermediate product transfer V^t at time t may naturally be taken as decision variables.

In the approach to production modelling taken here, it is essential to incorporate the relationship between the underlying technology and the information structures and constraints on the production units such that production policies may be formulated. The problem lies in how to pose reasonable models of technology and information structures such that their inter-relationship may be brought forth without too much complication. (The Team Theory of Marschak/Radner [1972] provide excellent examples of this type of endeavor.) Chapter 2 will consider, in a general setting, the effect of constraints on resource availability through formulations of laws of return under uncertainty. Chapter 3 proposes some special structured technologies so that production policies may be explicitly formulated.

1.9 Appendix

(1.9.1) Verifications of Axioms for Example (1.4.1):

Consider the function $\Phi : \mathbb{R}_+^n \times S \to \mathbb{R}_+$ defined by $\Phi(x,s) = Ae^s \prod_{i=1}^n x_i^{\alpha_i}$; $S = (-\infty,+\infty)$. For an arbitrary $\tilde{s} \in S$, the function $\Phi(\cdot,\tilde{s})$ is continuous, non-decreasing and homogenous on $\mathbb{R}_+^n$. Hence Axioms P1, P2 $\Longrightarrow$ P2.S, P3, P3.S, P5, P6 and P6.S follows immediately. Since A and α_i's are positive, $\Phi(\cdot,\tilde{s}) \geq 0$; so for some $\tilde{x} \in \mathbb{R}_+^n$, $\Phi(\tilde{x},\tilde{s}) =: \tilde{u} > 0$. Clearly, for all $s \in [\tilde{s},+\infty)$, $\Phi(\tilde{x},s) \geqq \tilde{u}$. Since the probability of the event $[s,+\infty)$ is positive, Axiom P4.1 holds. Axiom P4.2 and P4.2.I follow from the homogenity of $\Phi(\cdot,s)$ for all $s \in S$. Finally, to show P0 , consider an arbitrary $x \in \mathbb{R}_+^n$ and a closed set $F \subset \mathbb{R}_+$. Let $b(F)$ be the greatest lower bound of F , then $P_x^{-1}(F) = \{s \in S \mid P(x,s) \cap F \neq \emptyset\} = \{s \in S \mid \Phi(x,s) \geqq b(F)\}$. Since $\Phi(x,\cdot)$ is continuous and monotone on S , $P_x^{-1}(F)$ is a closed interval, hence measurable □

(1.9.2) Verification of Axioms for Example (1.4.2)

(P1) For each $s \in S$, $x = 0$ and $A(s)\cdot z \leqq x$ implies $z = 0$, consequently $B(s)\cdot z = 0$. Hence $P(x = 0 , s) = \{0\}$, $s \in S$.

(P2) For each $s \in S$ and $x \in \mathbb{R}_+^n$, (1.4.2.2) implies the set $\{z \in \mathbb{R}_+^K \mid A(s)\cdot z \leqq x\}$ is bounded. Then (1.4.2.3) implies $P(x,s)$ is bounded. Axiom P2.S follows from P2 since the output space is finite dimensional.

(P2.I) Consider an arbitrary fixed input $\tilde{x} \in \mathbb{R}_+^n$, (1.4.2.2) implies $z_k^* := \mathrm{Sup}_{s\in S} \left\{ \underset{A_{ik}(s)>0}{\mathrm{Min}} [\tilde{x}_i/A_{ik}(s)] \right\} < +\infty$, $k = 1,2, \ldots, K$. The z_k^*'s act as bounds on the feasible intensity of the activities. Then by

(1.4.2.3), for all $u \in P(\tilde{x},s)$, $u_j \leqq \sum_{k=1}^{K} B_{jk}(s)z_k^* \leqq M_j \cdot \sum_{k=1}^{K} z_k^* =: u_j^*$, $j = 1, \ldots, m$. Since the bounds u_j^* on outputs are constant over S , $P(\tilde{x},\cdot)$ is integrably bounded.

(P3, P3.S, P5, P6, P6.S) are trivially true since the technology is of linear activity type.

(P4.1) is merely a restatement of (1.4.2.4). P4.2 and P4.2.I follows from the constant return to scale of the technology.

(P0) Consider the following functions:

$$(s,z) \in S \times \mathbb{R}_+^K \to f_1(s,z) := (B(s),z) \ ;$$
$$(B,z) \in \mathbb{R}_+^{mK} \times \mathbb{R}_+^K \to f_2(B,z) := B \cdot z \ ;$$
$$(s,z) \in S \times \mathbb{R}_+^K \to g_1(s,z) := (A(s),z) \ ;$$
$$(A,z) \in \mathbb{R}_+^{nK} \times \mathbb{R}_+^K \to g_2(A,z) := A \cdot z \ .$$

By assumption (1.4.2.1) and the continuity of linear transforms, the functions $f := f_2 \circ f_1$ and $g := g_2 \circ g_1$ are $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+^K)$ measurable. Thus, for an arbitrary closed set F in $\mathbb{R}_+^M$, and $\tilde{x} \in \mathbb{R}_+^n$, $D_1 := \{(s,z) \in S \times \mathbb{R}_+^K \mid f(s,z) \in F\}$ and $D_2 := \{(s,z) \in S \times \mathbb{R}_+^K \mid g(s,z) \in \{y \in \mathbb{R}_+^n \mid y \leqq \tilde{x}\}\}$ are elements in $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+^K)$. Then $\{s \in S \mid P(\tilde{x},s) \cap F \neq \emptyset\} = \mathrm{Proj}_S \ [D_1 \cap D_2]$ belongs to $\mathcal{S}$ by an application of the classical projection theorem □

<u>(1.9.3) Verification of Axioms for Example (1.4.3)</u>

Axioms P1, P2, P2.I, P3, P3.S, P5, P6 and P6.S may be established as in (1.9.2). Since the output space is infinite dimensional, P2.S in general is not true.

(P4.1) Consider the first period. Since (1.4.2.4) holds for the output coefficient matrix $B_1(s_1)$, there exists an initial endowment $w \in \mathbb{R}_+^m$

and an input x (with a first period input $x^1 \in \mathbb{R}_+^m$ such that some output u with a positive i-th component in the first period (i.e., $(u^1)_i > 0$) is attainable with a positive probability. This verifies P4.1 since whatever happens after the first period does not affect the positivity of $(u^1)_i$.

(P4.2 and P4.2.I) May be established as in (1.9.2). It is remarked that P4.2 and P4.2.I are valid only because the initial endowment w is considered as exogenous input (in accordance with our viewpoint of an unconstrained technology). In decision models, this may not be reasonable anymore.

(PO) May be established by a straightforward modification of the verification offered in (1.9.2). The classical projection theorem applies since $(l_\infty)_+^n$ and $(l_\infty)_+^m$ are separable $\square$

CHAPTER 2

DYNAMIC LAWS OF RETURNS UNDER UNCERTAINTY

2.1 Introduction

The study of the effect of resource constraints (limitations) on production has always been an important topic in the theory of production. Turgot [1767] introduced into economic thought a proposition which has come to be known as the Law of Diminishing Returns at the intensive margin. The original spirit of the law was concerned with the restraint on agricultural production imposed by the scarcity of land. From this viewpoint, the particular formulation of the law in terms of diminishing product increments is non-essential to its significance. In bare form, the issue is: whether a bound on the inputs of a proper subset of factors leads to bounded outputs, when the other inputs may be applied in unlimited amounts. This more basic formulation of the law is described by Menger [1936] as an intersecting assertion.

Shephard [1970-b] gave a meta-economic proof of an intersecting assertion of the law of returns for production with scalar outputs, using the theoretical steady state framework of production introduced in Shephard [1967, 1970-a]. Subsequently, other formulations of the law (both steady state and dynamic, single and multiple products) had been offered along the same line. See Färe [1972, 1978, 1980], Shephard/Färe [1974] and Shephard/Färe [1980, Chapter 3].

This chapter extends the formulation of the laws of returns to allow for uncertainty in the production processes. This extension is

meaningful since the limitation of resource may be relevant only under certain production enviroment; depending, for instance, on the weather conditions, machine failures etc. Furthermore, the impact of resource limitation is contingent upon (a) the availability of substitutable resources; (b) the development of alternative production techniques; typically neither contingency is forseen with certainty.

The formulation of laws of return under uncertainty to be given brings out to a certain extent the inter-relationship between the underlying technology and the information structures of the producers. It is found that for production under uncertainty, the information structures play a role in limiting outputs, leading to a notion of diminishing returns in information.

2.2 Background: Essentiality and Limitationality

This chapter uses the axiomatic framework of stochastic production correspondences developed in Chapter 1. For simplicity of exposition, the output space is specialized to $(L_\infty)_+$ or $(l_\infty)_+$. The special case of deterministic production correspondences (see Remark (1.3.3)) is used freely when convenient. *Axiom* L6.S, *i.e., strong disposal of outputs, is assumed throughout and Axiom* L5 *(closure of graph) is stated with the weak* topology on the input space.*

The purpose of this section is to introduce the notions relevant to the formulation of the laws of returns. These notions are stated in a deterministic framework of production. The definitions used are

basically refinements of those originally used by Shephard [1970-b] and extended in the already cited references; so are the propositions which give the deduction of the deterministic laws of returns.

As indicated in the introduction, one is interested in the effect of the boundedness of input factors on the level of outputs attainable. In a dynamic framework, it is convenient, for production planning purposes, to specify the time periods over which the bounds on inputs are relevant. Let $I \subset\subset \{1, \ldots, n\}$ denote a proper subset of the n input factors. The time period (support) over which an input factor $i \in I$ is limited is taken as an element $S_i \in \Sigma_i$ (Σ_i being the σ-field on $\mathbb{R}_+$ for the i-th input history, see Shephard/Färe [1980] or Chapter 1). Collectively for the factor group I, the relevant time periods (supports) of the input-bound is denoted by a family

$$SI \equiv \{S_i \; ; \; S_i \in \Sigma_i \; , \; i \in I\} \; . \tag{2.2.1}$$

Focusing attention to the support SI, define for a vector input history $x \in (L_\infty)_+^n$

$$x_{SI} := (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in (L_\infty)_+^n$$

$$\text{where} \quad \tilde{x}_i(t) := \begin{cases} 0 & \text{if} \quad t \notin S_i \quad \text{or} \quad i \notin I \; ; \\ x_i(t) & \text{if} \quad t \in S_i \quad \text{and} \quad i \in I \; . \end{cases} \tag{2.2.2}$$

The partial ordering of vector inequality on the inputs may be restricted to apply only on the support SI :

$$\text{(2.2.3)} \qquad \begin{array}{l} \text{for } x, y \in (L_\infty)^n_+ , \; x_{SI} \leqq y_{SI} \text{ if and} \\ \text{only if } (\tilde{x}_1, \ldots, \tilde{x}_n) \leqq (\tilde{y}_1, \ldots, \tilde{y}_n) . \end{array}$$

The "truncated" input history x_{SI} defined in (2.2.2) may be interpreted alternatively as a subvector, i.e., as a component of the original vector x. With this notation, a bound on an input factor group I over a support SI may be modelled by restricting inputs to the set

$$\text{(2.2.4)} \qquad \left\{x \in (L_\infty)^n_+ \mid x_{SI} \leqq x^o_{SI}\right\} , \; x^o \in (L_\infty)^n_+ , \; x^o_{SI} \geq 0$$

where the reference subvector x^o_{SI} acts as a bound on the inputs.

Similarly, let SO (which is an element of the σ-field of the single output history) specify the time period (support) over which limitations on outputs are relevant. The output restricted to SO, denoted u_{SO}, is defined as in (2.2.2) with an analogous definition of a partial ordering (2.2.4), and the interpretation as subvectors.

A remark is in order concerning the relationship between SI and SO. Since future inputs cannot affect past outputs, it is necessary to postulate that

$$\text{(2.2.5)} \qquad \begin{array}{l} \text{Inf } \{t \in \mathbb{R}_+ \mid t \in S_i \text{ for some } i \in I\} \leqq \text{Inf } \{t \in \mathbb{R}_+ \mid t \in SO\} ; \\ \text{Sup } \{t \in \mathbb{R}_+ \mid t \in S_i \text{ for some } i \in I\} \leqq \text{Sup } \{t \in \mathbb{R}_+ \mid t \in SO\} . \end{array}$$

For SI and SO satisfying (2.2.5), the following formulation of a law of returns is meaningful:

<u>(2.2.6) Definition</u>: An input factor group $I \subset\subset \{1, \ldots, n\}$ over support SI is *weak-limitational* for outputs over support SO if for

every output (reference) subvector $u^o_{SO} > 0$, there exists an input bound $x^o_{SI} \geq 0$ such that $L(u) \cap \left\{x \in (L_\infty)^n_+ \mid x_{SI} \leqq x^o_{SI}\right\}$ is empty for all output u with $u_{SO} \geqq u^o_{SO}$.

(2.2.7) Definition: An input factor group $I \subset\subset \{1, \ldots, n\}$ over support SI is *essential* for outputs over support SO if for every output u with $u_{SO} > 0$ and $L(u)$ not empty, $L(u) \cap \left\{x \in (L_\infty)^n_+ \mid x_{SI} = 0\right\}$ is empty.

(2.2.8) Proposition: For an input factor group $I \subset\subset \{1, \ldots, n\}$ over support SI to be weak-limitational over support SO , it is necessary and sufficient that (I,SI) is essential for SO .

A proof of this proposition may be found in Shephard/Färe [1980, Chapter 3, Proposition (3.3-1)].

It is conceivable that although (I,SI) is essential for output on SO and x_{SI} is bounded by some x^o_{SI} , via appropriate factor and time substitution, any bound u^o_{SO} on the output may be exceeded. For example, inputs (I,SI) may be needed only to initiate a new production process which otherwise does not require input factors I in its subsequent operations. To formulate a stronger version of limitationality, define

(2.2.9) Definition: An input factor group $I \subset\subset \{1, \ldots, n\}$ over support SI is *strong-limitational* for outputs over support SO if for every input bound $x^o_{SI} \geq 0$ there exists a bound $u^o_{SO} > 0$ on the output such that $L(u) \cap \left\{x \mid x_{SI} \leqq x^o_{SI}\right\}$ is empty for all output u with $u_{SO} \geqq u^o_{SO}$.

In the steady state framework of production, Färe [1972, 1980] and Shephard/Färe [1974] gave a sufficient condition for strong limitationality. In the following, a further sufficient condition for strong-limitationality is given via a regularity assumption on the scaling of production.

With a single output history, a dynamic production correspondence $P : (L_\infty)^n_+ \to 2((L_\infty)_+)$ may be represented by the following functionals:

(2.2.10)
$$\begin{aligned} &\text{For given } w \in (L_\infty)_+ \text{ with } \|w\| = 1 , \\ &\Phi(x \mid w) := \text{Max } \{\alpha \in \mathbb{R}_+ \mid \alpha \cdot w \in P(x)\} , \quad x \in (L_\infty)^n_+ . \end{aligned}$$

Note that because of Axiom L5, $\Phi(\cdot \mid w)$ is well defined for each w ; furthermore, by L4.2, either $\Phi(\cdot \mid w) \equiv 0$ or $\Phi(\cdot \mid w)$ has the range $[0,+\infty)$. In a sense, $\Phi(\cdot \mid w)$ is a production functional which gives the maximal scale of an output "time-pattern" w attainable.

(2.2.11) Definition: A dynamic production structure $P : (L_\infty)^n_+ \to 2((L_\infty)_+)$ satisfies *regular scaling* if

(a) there exists a $B \in \mathbb{R}_{++}$ such that for all $x \in (L_\infty)^n_+$ and all output pattern $w \in (L_\infty)_+$, $\|w\| = 1$, defining

$$\sigma^*_x(w) := \begin{cases} 0 \ \text{ if } \ \Phi(\alpha \cdot x \mid w) = 0 \ \text{ for all } \ \alpha \in \mathbb{R}_{++} ; \\ \text{Inf } \{\sigma > 0 \mid \Phi(\sigma \cdot x \mid w) > 0\} \ \text{ if otherwise;} \end{cases}$$

it is true that $\Phi(\sigma^*_x(w) \cdot x \mid w) \leq B$; and

(b) for every $\lambda \in \mathbb{R}_{++}$, there exists a $\theta_\lambda \in \mathbb{R}_{++}$ such that for each $w \in (L_\infty)_+$, $\|w\| = 1$, the following functional inequality holds:

(2.2.12) $\Phi(\lambda \cdot x \mid w) \leq \theta_\lambda \cdot \Phi(x \mid w)$ for all $x \in (L_\infty)^n_+$ with $\Phi(x \mid w) > 0$.

It is important to note that regular scaling allows the output sets $P(\lambda \cdot x)$ to span different output time patterns as λ changes; as would be the case where increased inputs allows for more production possibilities.

Regular scaling appears to be a rather mild and reasonable regularity condition on a production technology. The class of functionals which satisfy the functional inequality (2.2.12) appears to be rather large. Clearly, homogenous and sub-homogenous functionals satisfy (2.2.12). Furthermore, super-homogenous functionals like : $= \Phi(\lambda x \mid w) = \lambda^k \cdot \Phi(x \mid w)$ with $k > 1$; also satisfy (2.2.12). In fact, the postulate of regular scaling was inspried by Eichhorn [1968] which used the homogeneity of production functions to derive the law of diminishing incremental return over the whole range of inputs. This class of functionals is characterized in Mak [1980-b]. Here, regular scaling is used to establish:

(2.2.13) Proposition: Suppose a production structure satisfies regular scaling, then an input factor group $I \subset\subset \{1, \ldots, n\}$ over support SI is strong-limitational for outputs over support SO if (I,SI) is essential for outputs over SO .

The reasoning underlying this proposition is actually very simple : - if an input bound x^o_{SI} does not bound outputs on SO , factors I on SI must be "infinitely substitutable" by the other factors on SI and other input supports. Then because of the assumed boundedness of the efficient subsets (see the asymmetric axiom E in Section 1.6), this is possible only if (I,SI) is not essential for outputs on SO .

Proof of Proposition (2.2.13)

Suppose (I,SI) is not strong-limitational for SO . Then there exists an input bound $x^o_{SI} \geq 0$ such that for all output bound $u^o_{SO} > 0$, there is an input x with $x_{SI} \leqq x^o_{SI}$ and an output $u \in P(x)$ with $u_{SO} \geqq u^o_{SO}$. In particular, consider a sequence of output bounds $\left\{v^k := \alpha^k \cdot 1_{SO}\right\}$ where $\alpha^k \geqq 1$ and $\{\alpha^k\} \to +\infty$. Let $\{x^k\}$ and $\{u^k\}$ be sequences of inputs and outputs such that $x^k_{SI} \leqq x^o_{SI}$, $u^k \in P(x^k)$ and $u^k_{SO} \geqq v^k$. Considering the indicator function 1_{SO} as an output history, it follows from L3.S that $x^k \in L(1_{SO})$ for all k. Define an infinite sequence of scalars by

$$\gamma^k := \min \left\{\alpha \in \mathbb{R}_+ \mid \alpha \cdot x^k \in L(1_{SO})\right\}, \ k = 1,2, \ldots .$$

The γ^k's are well defined because of Axiom L5 and L4.2.

Claim: There exists a scalar $K \geq 1$ such that $\Phi(\gamma^k \cdot x^k \mid 1_{SO}) \leqq K$ for all index k. To prove this claim, first note that by construction, $\Phi(\gamma^k \cdot x^k \mid 1_{SO}) \geqq 1$. If $\Phi(\gamma^k \cdot x^k \mid 1_{SO}) = 1$ for all k, then the claim is trivially true by taking K to be 1. So, let $\Phi(\gamma^k \cdot x^k \mid 1_{SO}) > 1$ for some indices. Consider arbitrarily such an index k. Clearly, either there exists $\theta \in (0,1)$ with $\Phi(\theta\gamma^k \cdot x^k \mid 1_{SO}) \in (0,1)$; i.e., $\gamma^k \cdot x^k$ is a point of discontinuity of $\Phi(\cdot \mid 1_{SO})$ along the ray $\{\lambda \cdot x^k \mid \lambda \geq 0\}$ but not a first jump point to a positive scaling of 1_{SO} ; or $\gamma^k \cdot x^k$ is in fact a first jump point.

Now use contra-positive argument: Suppose there does not exist a positive scalar K such that $\Phi(\gamma^k \cdot x^k \mid 1_{SO}) \leqq K$ for all k. Then there is an infinite subsequence $\{\gamma^j \cdot x^j\} \subset \{\gamma^k \cdot x^k\}$, such that $\left\{\Phi(\gamma^j \cdot x^j \mid 1_{SO})\right\}$ diverges to $+\infty$. If in the sequence $\{\gamma^j \cdot x^j\}$, there

is an infinite subsequence of first jump points, then condition (2.2.11a) is violated. On the other hand, if such a subsequence does not exist, one may as well assume none of the points $\gamma^j \cdot x^j$ is a first jump point. Fix an arbitrary $\lambda^* \in (0,1)$. Since $\gamma^j \cdot x^j$ is not a first jump point, there exists an input vector $z^j \in [\lambda^* \gamma^j \cdot x^j , \gamma^j \cdot x^j)$ such that $1 > \Phi(z^j \mid 1_{SO}) > 0$. Consider the sequence $\{z^j\}$ thus chosen. Because $\lambda^* \in (0,1)$, $\gamma^j \leq 1$ for all j by definition, and $\Phi(\cdot \mid 1_{SO})$ is monotone along rays (Axiom L.3),

$$\frac{\Phi(z^j/\lambda^* \mid 1_{SO})}{\Phi(z^j \mid 1_{SO})} \geq \frac{\Phi(\gamma^j \cdot x^j \mid 1_{SO})}{\Phi(z^j \mid 1_{SO})} \geq \Phi(\gamma^j \cdot x^j \mid 1_{SO}) \xrightarrow{j} +\infty .$$

Hence condition (2.2.11b) does not hold for the scaling factor $1/\lambda^*$. This contradicts the hypothesis of regular scaling, thus establishing the claim.

Next, it is shown that $\text{Inf}\ \{\gamma^k\} = 0$. Suppose otherwise, then there exists a $\varepsilon > 0$ such that $\gamma^k \geq \varepsilon$ for all k . Then by the monotonicity of $\Phi(\cdot \mid 1_{SO})$ along rays and the hypothesis of regular scaling

$$0 < \Phi(x^k \mid 1_{SO}) = \Phi\left(\frac{1}{\gamma^k}\cdot\gamma^k\cdot x^k \mid 1_{SO}\right) \leq \Phi(\tfrac{1}{\varepsilon}\cdot\gamma^k\cdot x^k \mid 1_{SO})$$
$$\leq \theta_{1/\varepsilon}\cdot\Phi(\gamma^k\cdot x^k) \leq \theta_{1/\varepsilon}\cdot K < +\infty$$

where K is the bound on $\left\{\Phi(\gamma^k\cdot x^k \mid 1_{SO})\right\}$ established in the earlier claim. This contradicts the original assumption that $\left\{\Phi(x^k \mid 1_{SO})\right\} \to +\infty$.

Since $\text{Inf}\ \{\gamma^k\} = 0$, there is a subsequence $\{\gamma^p\} \subset \{\gamma^p\}$ with $\{\gamma^p\} \downarrow 0$. For each index p , $\gamma^p\cdot x^p \in L(1_{SO})$, hence an input

$z^p \leqq \gamma^p \cdot x^p$ may be constructed such that $z^p \in E(1_{SO})$ (see the proof of Proposition (1.6.4)). By construction, $z^p_{SI} \leqq \gamma^p \cdot x^p_{SI} \leqq \gamma^p \cdot x^p_{SI}$. Since $\{\gamma^p\} \downarrow 0$, it is seen that

$$\operatorname{Inf} \left\{ \|x - y\| \mid x \in \overline{E(1_{SO})^*} , y_{SI} = 0 \right\} = 0 .$$

Then, using the same argument as for the proof of Proposition (2.2.8), one may show that $\overline{E(1_{SO})^*} \cap \{x \mid x_{SI} = 0\}$ is not empty. Since according to L5, $\overline{E(1_{SO})^*} \subset L(1_{SO})$; and because of Fact (1.6.5), (I,SI) cannot be essential for SO , completing the contra-positive proof □

2.3 Laws of Returns Under Uncertainty

As in the deterministic case, the issue is whether a bound on some input factors will limit outputs. In a deterministic model of production, since every feasible input-output combination (x,u) with $x \in L(u)$ can be regarded as a production policy, laws of returns may be deduced strictly from the properties of the deterministic technology. With the introduction of uncertainty, the actual process of production is no longer completely characterized by the underlying stochastic technology. The attainability of (or the limitations on) outputs is a consequence of the production policies of the producers. Thus, there is the question of the role played by information structures on the laws of returns. Furthermore, since the inputs and outputs associated with the production policies in general depend on the state of environment, the notions of boundedness of inputs and outputs has to be clarified.

The consistency requirement (1.8.3) clearly indicates that broadly

speaking, an information structure acts as a constraint on the choice of production policies. In this sense, it may be taken as a bound on the information available for formulating production policies.

One may be interested only in the information available at certain time periods; for instance, it is important to know the availability of new production techniques when investment decisions have to be made on new plant capacities. Consider a subset $T \subset \{1,2, \ldots\}$ of decision time points; and let

$$\mathcal{J}_T^o \equiv \left\{ \mathcal{J}_t^o \right\}_{t \in T}$$

denote the "restriction" of a particular information structure $\mathcal{J}^o$ to T . A bound on information may be formulated by using the partial ordering:

(2.3.1) $\mathcal{J}_T \precsim \mathcal{J}_T^o$ if $\mathcal{J}_t^o$ is as fine as $\mathcal{J}_t$ for all $t \in T$.

When the information structure $\mathcal{J}$ of a producer satisfies $\mathcal{J}_T \precsim \mathcal{J}_T^o$, his information is said to be limited by $\mathcal{J}_T^o$ over the time period T . Later on, the special case of "perfect information" $\mathcal{J}_T^p$ defined by:

$$I_t \text{ is a singleton for all } I_t \in \mathcal{J}_t^p \quad (t \in T)$$

is useful. Note that every information structure $\mathcal{J}$ has $\mathcal{J}_T \precsim \mathcal{J}_T^p$; i.e., bounded by $\mathcal{J}_T^p$ over T .

For the input and output histories, it is clear that the input support SI (for factors I) and output support SO may be defined as in the deterministic case; and the (functions of) subvectors x_{SI} , u_{SO}

have the same meaning as in (2.2.2). However, since the bounds in general will depend on the states, bounds x^o_{SI} and u^o_{SO} should be taken as functions (recalling the notation of (1.8.4)):

$$\begin{aligned} & s \in S \to \underline{x}^o_{SI}(s) \; ; \\ (2.3.2) \qquad & \\ & s \in S \to \underline{u}^o_{SO}(s) \; ; \end{aligned}$$

where for each state $s \in S$, the partial ordering (2.2.3) of subvectors is valid. For simplicity, henceforth $\underline{x}^o_{SI}(\cdot)$ will be taken to be a constant function (denoted by its function value x^o_{SI}) when an input bound is to be specified for the formulation of the laws of returns.

With the definition of the support T, SI and SO ; bounds $\mathcal{J}^o_T$, $\underline{x}^o_{SI}(\cdot)$ and $\underline{u}^o_{SO}(\cdot)$; and the notion of production policies $(s \in S \longmapsto (\underline{x}(s),\underline{u}(s))$, the following *notion of limitationality on outputs* is meaningful for a state $s^* \in S$:

(2.3.3) there exists an output bound $u^o_{SO} > 0$ such that limited by the information $\mathcal{J}^o_T$, there does not exists an information structure $\mathcal{J}$ with $\mathcal{J}_T \precsim \mathcal{J}^o_T$ and an associated production policy $s \longmapsto (\underline{x}(s),\underline{u}(s))$ with $\underline{x}_{SI}(s) \equiv (\underline{x}(s))_{SI} \leqq x^o_{SI}$, all $s \in S$; and the output at state s^* is such that $\underline{u}_{SO}(s^*) \equiv (\underline{u}(s^*))_{SO} \geqq u^o_{SO}$.

Corresponding to the notion of strong and weak limitationality in the deterministic case, laws of returns under uncertainty may be formulated by either one of the following:

F1 For each bound $x^o_{SI} \geq 0$ on (I,SI) and each bound $\mathcal{J}^o_T$ on information, (2.3.3) holds.

F2 There exists a bound $x^o_{SI} \geq 0$ on (I,SI) such that for each bound $\mathcal{J}^o_T$ on information, (2.3.3) holds.

F3 For each bound $\mathcal{J}^o_T$ on information, there exists a bound $x^o_{SI} \geq 0$ on (I,SI) such that (2.3.3) holds.

F4 There exists a bound $\mathcal{J}^o_T$ on information such that for each bound $x^o_{SI} \geq 0$ on (I,SI), (2.3.3) holds.

F5 For each bound $x^o_{SI} \geq 0$ on (I,SI), there exists a bound $\mathcal{J}^o_T$ on information such that (2.3.3) holds.

F6 There exists a bound $x^o_{SI} \geq 0$ on (I,SI) and a bound $\mathcal{J}^o_T$ on information such that (2.3.3) holds.

For each of the above formulations, the consequence of the bound on output $\underline{u}(s^*)$ due to the bounds x^o_{SI} and $\mathcal{J}^o_T$ may be interpreted as follows : - irrespective of possible increased applications of other inputs, usage of better (finer) information at other times, and their associated policies, the output bound u^o_{SO} cannot be exceeded if the true state of nature is s^*. Hence, the increased application of the other input factors and the solicitation of finer information jointly have, in the sense of Menger's intersecting assertion, diminishing returns.

Note that F1 to F6 are related by:

$$\begin{tikzcd} & F2 \arrow[r] & F3 \arrow[dr] & \\ F1 \arrow[ur] \arrow[dr] & & & F6 \\ & F4 \arrow[r] & F5 \arrow[ur] & \end{tikzcd}$$

The subtle interplay between the bounds on information and the bounds on inputs will be made clear later by the characterization of F1 to F6. Off-handedly, by observing that $\mathcal{J}_T^o$ may be taken as fine as the perfect information $\mathcal{J}_T^p$ and x_{SI}^o may be arbitrarily large, one may see intuitively that in F2 and F3, it is the bounds on inputs; while in F4 and F5 it is the bound on information; which plays the key role in limiting outputs.

To give alternative characterization of F1 to F6, the following will be assumed for convenience:

(A.1) output support SO is a finite subset of $\{1,2, \ldots\}$.

As a consequence of (A.1) and the reasoning of (2.2.5), supports T and SI are also finite.

With respect to production policies, the following condition is imposed:

(A.2) If there is perfect information at all times (i.e., $T \equiv \{1,2,\ldots\}$ and $\mathcal{J} \equiv \mathcal{J}_T^p$) , then for each state $s \in S$ and input-output pair (x_s,u_s) with $u_s \in P(x_s,s)$, there exists a production policy with $\underline{x}(s) = x_s$ and $\underline{u}(s) = u_s$.

The above condition merely asserts that with perfect information, a producer may plan for production as in the deterministic case. The next two definitions address the case where perfect information is not available.

<u>(2.3.4) Definition</u>: Two states s' and s" are *indistinguishable* under $\mathcal{J}_T^o$ if for each $t \in T$ there is an element I_t in $\mathcal{J}_T^o$ which

contains both s' and s" .

<u>(2.3.5) Definition</u>: Information on s' , s" $\in$ S is *non-discriminating* for outputs over SO if for each $\mathcal{J}_T^o$ under which s' and s" are indistinguishable there exists a bound $B \in \mathbb{R}_{++}$ such that for all production policies consistent with $\mathcal{J}_T^o$, $\| \underline{u}_{SO}(s') - \underline{u}_{SO}(s") \| \leqq B$.

The above definition models the situation where the lack of information so restricts the choice of production policies that the resulting outputs may not be of arbitrarily large difference.

With all the preliminary notions defined, the following proposition gives the characterization of F1 to F6:

<u>(2.3.6) Proposition</u>: Suppose a stochastic production technology satisfies the stochastic weak axioms (1.4.6), E and L6.S, and Assumptions (A.1) and (A.2) are enforced. Consider a state $s^* \in S$:

(i) If the correspondence P_{s^*} (see 1.2.3) satisfies regular scaling (2.2.11), then F1 is equivalent to : - (I,SI) is essential for SO under s^* ; i.e., (2.2.7) holds for the correspondences P_{s^*} .

(ii) F2 is equivalent to : - (I,SI) is essential for SO under s^* .

(iii) F3 is equivalent to : - for each bound $\mathcal{J}_T^o$ on the information, there exists a state s' which is indistinguishable from s^* under $\mathcal{J}_T^o$, information on s' and s^* is non-discriminating for SO , and (I,SI) is essential for SO under s' .

(iv) Suppose the correspondences P_s satisfy regular scaling for each $s \in S$, then F4 is equivalent to: there exists a bound $\mathcal{J}_T^o$ on information under which there exists a state s' indistinguishable from s^* , information on s' and s^* is non-discriminating for

SO , and (I,SI) is essential for SO under s' .

(v) F5 (F6) is equivalent to : - for every bound $x^o_{SI} \geq 0$ on (I,SI)

(vi) (there exists a bound $x^o_{SI} \geq 0$ on (I,SI)), there exists a bound $\mathcal{J}^o_T$ on information under which exists a state s' indistinguishable from s^* , information on s' and s^* is non-discriminating for SO and there exists an output bound u^o_{SO} such that $u \notin P(x,s')$ for all input x with $x_{SI} \leqq x^o_{SI}$ and output u with $u_{SO} \geqq u^o_{SO}$.

__Proof__:

(i) Suppose (I,SI) is essential for SO under s^* and P_{s^*} satisfies regular scaling, then by Proposition (2.2.13) there exists for each input bound x^o_{SI} an output bound $u^o_{SO} > 0$ such that $x_{SI} \leqq x^o_{SI}$ implies $u \notin P(x,s^*)$ if $u_{SO} \geqq u^o_{SO}$. Then F1 follows from the technical feasibility condition (1.8.4). To show the converse, suppose (I,SI) is not essential for SO under s^* . Then there is an input $\bar{x}$ with $\bar{x}_{SI} = 0$ and output $\bar{u} \in P(\bar{x},s^*)$ with $\bar{u}_{SO} > 0$. Then by the scaling Axiom L4.2 and (A.1), for every bound $u^o_{SO} > 0$ there is a scalar θ such that $(\theta \cdot \bar{u})_{SO} \geqq u^o_{SO}$ and a scalar λ_θ such that $\theta \cdot \bar{u} \in P(\lambda_\theta \cdot \bar{x},s^*)$. Finally, with $\mathcal{J}^o_T$ taken as the perfect information $\mathcal{J}^P_T$, (A.2) implies F1 does not hold.

(ii) May be established as (i) using Proposition (2.2.8).

(iii) Suppose (I,SI) is essential for SO under s' . Then by (1.8.4) and Proposition (2.2.8), there exists a bound $x^o_{SI} \geq 0$ on (I,SI) and a bound $u^o_{SO} > 0$ such that every production policy has $\underline{u}(s') \ngeq u^o_{SO}$. If s' is indistinguishable from s^*

and information on s' and s^* is non-discriminating for outputs on SO , then (F3) follows directly. Conversely, assume contra-positively that there is an information bound $\mathcal{J}_T^o$ under which no state s' which is indistinguishable from s^* satisfies either: (a) (I,SI) essential for SO under s' ; or (b) information on s' and s^* is non-discriminating for SO . If such an $\mathcal{J}_T^o$ exists, clearly one may take $\mathcal{J}_T^o \equiv \mathcal{J}_T^p$. Since s^* is indistinguishable from itself and s^* and itself is non-discriminating for SO , (I,SI) is not essential for SO under s^* . Then as argued in the latter part of (i), (F3) does not hold. (iv), (v) and (vi) may be established using similar arguments. In (iv), Proposition (2.2.13) is used to establish the existence of an output bound. In (v), the existence of output bound is assumed outright. (v) is a weaker assertion than (iv) merely because if the input bound x_{SI}^o is relaxed (i.e., made larger), a less fine $\mathcal{J}_T^o$ bound on information may be needed to locate a state s' under which output is limited □

Formulation F1 to F6, of course, are not the only possible formulations of the laws of returns under uncertainty. In fact, they are the simplest formulations possible. Formulations may be extended to the cases: (a) bounds are expressed in terms of norms; (b) output bounds u_{SO}^o are taken as functions on S instead of focusing attention on a state s^* ; (c) input bounds x_{SI}^o are taken as functions; etc. However, it is hoped that the formulation in this section has succeeded in indicating the complexity of dynamic production under uncertainty, in particular the interplay between information, technology and production decisions.

CHAPTER 3

STOCHASTIC HOMOTHETIC PRODUCTION CORRESPONDENCES

3.1 Introduction

The modelling of an actual production technology involves considerable trade-off: on the one hand, the model must be sophisticated enough to capture the relevant production phenomenon of interest; on the other hand, the model must be of manageable complexity. When uncertainty is involved, the task of modelling is more difficult since one has to contend with the influences of the uncertain production environment on the production processes.

This chapter uses two ideas that are quite often used in economic literature as the key to the formulation of some simple but yet reasonable stochastic models of technology. The first one is the notion of scaling of production: that production of one level of outputs is related to the production at another level. The second one is transformation: the relevant production technology under some production environment being in some sense a transformation of the technology under another environment. These two ideas are integrated via a generalized notion of scaling (or input and output factors) which is subsequently used to yield stochastic homothetic production correspondences.

The form of a stochastic homothetic production correspondence leads quite naturally to some further special structures which afford rather simple representations. Through these representations, production planning under uncertainty is seen to be possible: firstly, in the case of overall planning of production in conjunction with the notion of

confidence indexed production correspondences; secondly, in conjunction with a model of information, the optimal policy of a simple production system is shown to exist, much similar to the models of multi-stage stochastic programming in Operations Research.

3.2 Scaling and Transformation of Factors of Production

This section develops a generalized notion of scaling of inputs and outputs as a background for subsequent exposition. To avoid diversion from the main topic of stochastic production, the proofs for some of the propositions will not be given here. They may be found in a forthcoming paper (Mak [1981]).

The definition of scaling of factors of production will be given in terms of an input space X. The corresponding definition for the case of an output space U is identical to that for X.

(3.2.1) Definition: A mapping $T : \mathbb{R}_+ \times X \to X$ is a *scaling operation* on the space X if it satisfies:

(i) For each $\mu \in \mathbb{R}_{++}$, $T(\mu,\cdot) : X \to X$ is a one-one and onto map; for each $x \in X$, $x \neq 0$, $T(\cdot,x) : \mathbb{R}_+ \to X$ is a one-one map.

(ii) $T(1,x) = x$ and $T(0,x) = 0 = T(\mu,0)$ for all $\mu \in \mathbb{R}_+$, all $x \in X$.

(iii) For all $\mu \in \mathbb{R}_{++}$, $T(\mu,x) = y$ if and only if $T(1/\mu, y) = x$.

(iv) For all $(\lambda,\mu) \in \mathbb{R}^2_{++}$, $T(\lambda\mu,x) = T(\lambda,T(\mu,x))$.

It should be noted that the above set of properties are not independent; explicitly, (3.2.1-ii) and (3.2.1-iv) implies (3.2.1-iii).

Given a scaling operation T on an input space X, an input vector y is called a *scaled version* of an input x, denoted $y \mathcal{R} x$,

if there is a scalar $\mu \in \mathbb{R}_{++}$ with $T(\mu,x) = y$. The relationship $\mathcal{R}$ thus induced by T clearly satisfies: (a) $x\mathcal{R}x$, by (3.2.1-ii); (b) $y\mathcal{R}x$ iff $x\mathcal{R}y$, by (3.2.1-iii); and (c) $z\mathcal{R}y$ and $y\mathcal{R}x$ implies $z\mathcal{R}x$, by (3.2.1-iv). Hence, $\mathcal{R}$ generates equivalence classes of scaled versions of vectors. Denote the partition of the input space X via such equivalence classes by $\mathcal{D} := \{D_\beta\}_{\beta \in B}$. The index set B may be taken as a collection of representative elements, one from each equivalence class. If $\bar{x} \in B$, then $D_{\bar{x}}$ is simply the set $\{x \in X \mid x\mathcal{R}\bar{x}\}$. The singleton $\{0\}$ belongs to $\mathcal{D}$. All these should be clear from the usual (radial) scaling of input vectors:

$$(\mu,x) \in \mathbb{R}_+ \times X \mapsto T(\mu,x) := \mu \cdot x \ .$$

Here, $B := \{x \in X \mid \|x\| = 1\} \cup \{0\}$; and for $\bar{x} \in B$, $D_{\bar{x}}$ is the ray $\{\mu \cdot \bar{x} \mid \mu \in \mathbb{R}_{++}\}$.

For simplicity, scaling operations will henceforth be denoted by symbols $*$ or $\circledast$ so as to be distinguished from the usual radial scaling (denoted $\cdot$). For instance, for a scaling operation $(T,*)$ on X , $\mu * x \equiv T(\mu,x)$ for all $(\mu,x) \in \mathbb{R}_+ \times X$.

An operator $\odot$ (Shur operator) may be defined on an input space $X \equiv (L_\infty)^n_+$ as follows: for all x , $y \in (L_\infty)^n_+$, $w = x \odot y$ if and only if $w = (w_1, \ldots, w_n)$ where $w_i(t) = x_i(t) y_i(t)$ for every $t \in \mathbb{R}_+$, $i \in \{1, \ldots, n\}$. A similar definition applies to $X \equiv (l_\infty)^n_+$. With this definition, a restricted class of scaling operations is introduced:

<u>(3.2.2) Definition</u>: A scaling operation $(T,*)$ on an input space $X \equiv (l_\infty)^n_+$ is *normal* if it is representable as:

$$(3.2.2.1) \qquad \mu * x = T(\mu,x) := \Delta(\mu,x) \circledast x \ , \quad (\mu,x) \in \mathbb{R}_+ \times X \ ;$$

where $\Delta : \mathbb{R}_+ \times X \to X$ and $\circledast$ is a Shur operator.

Note that if $(T,*)$ is a normal scaling operation on $x \equiv (L_\infty)_+^n$, then for each input $x \in (L_\infty)_+^n$ with $x_i(t) = 0$, $t \in \mathbb{R}_+$, the i-th component of every scaled version of x is also null at t; i.e., $(\mu * x)_i(t) = 0$ for all $\mu \in \mathbb{R}_{++}$. Clearly, the usual radial scaling is normal. The adjective normal refers to the hypothesis that a null component of an input cannot be rendered non-null by scaling.

Suppose $(T,*)$ is a normal scaling operation on X. Then the transitivity condition (3.2.1-iv) on T and (3.2.2.1) together require the mapping to satisfy the following functional equation (compare with Färe [1973, equation Γ.4]):

$$(3.2.3) \quad \Delta(\lambda\mu,x) = \Delta(\lambda,\Delta(\mu,x) \circledast x) \circledast \Delta(\mu,x) \ , \quad (\lambda,\mu,x) \in \mathbb{R}_{++}^2 \times X \ .$$

To solve this functional equation, it is convenient to define:

<u>(3.2.4) Definition</u>: A mapping $F : X \to X$ on an input space X is *reversible* if either (a) F is invertible; or (b) F is onto and for all x, $y \in X$, $F(x) = F(y)$ implies $F(\mu \cdot x) = F(\mu \cdot y)$ for all $\mu \in \mathbb{R}_+$.

For a reversible mapping F on X, define a *reverse* $\overleftarrow{F}$ as follows: for each $x \in X$, let $\overleftarrow{F}(x)$ take an arbitrarily fixed value w with $F(w) = x$. Clearly, if F is invertible, then $\overleftarrow{F}$ is the usual inverse function. Otherwise, many $\overleftarrow{F}$ are possible.

<u>(3.2.5) Proposition</u>: Suppose a mapping $F : X \to X$, $F(x) \equiv (F_1(x), \ldots, F_n(x))$, on an input space X satisfies: (i) $F(0) = 0$; (ii) F is reversible; (iii) for all $x \in X$, $(F_i(x))(t) = 0$ implies $(F_i(\mu \cdot x))(t) = 0$ for all $\mu \in \mathbb{R}_+$, $i = 1, \ldots, n$ and $t \in \mathbb{R}_+$. Then an operation $*$ on X defined by:

$$(3.2.5.1) \qquad \mu * x := F(\mu \cdot \overleftarrow{F}(x)) \ , \ (\mu,x) \in \mathbb{R}_+ \times X$$

where $\overleftarrow{F}$ is an reverse of F , is a normal scaling operation on X . Furthermore,

$$(3.2.5.2) \qquad \Delta(\mu,x) := F(\mu \cdot \overleftarrow{F}(x)) \circledast x^{-1} \ , \ (\mu,x) \in \mathbb{R}_+ \times X$$

is a solution of the functional equation (3.2.4); where $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$ is defined by $x_i^{-1}(t) := 1/x_i(t)$, $i = 1, \ldots, n$; with the convention that $1/0 \equiv 1$.

To establish the converse to Proposition (3.2.5), a further regularity condition on the scaling operation is needed:

<u>(3.2.6) Definition</u>: A scaling operation T on an input space X is said to satisfy the *size condition* if there is a scalar $K > 0$ such that every equivalence class D_β (induced by T via the relation of scale versions) not equal to $\{0\}$ contains an input x with $\|x\| = K$.

<u>(3.2.7) Proposition</u>: Suppose a normal scaling operation $(T,*)$ on an input space X satisfies the size condition. Then there is a mapping $F : X \to X$ satisfying the conditions (3.2.5-i), (3.2.5-ii) and (3.2.5-iii) such that for all $(\lambda,x) \in \mathbb{R}_+ \times X$, $\lambda * x = F(\lambda \cdot \overleftarrow{F}(x))$.

Propositions (3.2.5) and (3.2.7) together state that under regularity conditions (3.2.5-i, ii, iii) and (3.2.6), there is a

natural association between normal scaling operations and reversible mappings (transformations), generalizing Färe [1973, Theorem 1]. This fact will serve as a basis for constructing general homothetic production structures in the subsequent sections. In particular, the following simple fact will be useful:

<u>(3.2.8) Proposition</u>: Suppose a normal scaling operation $*$ and a reversible transformation $F : X \to X$ on an input space are related in the sense that $\lambda * x = F(\lambda \cdot \overset{\leftarrow}{F}(x))$ for all $(\lambda, x) \in \mathbb{R}_+ \times X$ and some reverse $\overset{\leftarrow}{F}$ of F. Then $F(\lambda \cdot w) = \lambda * w$ for all $(\lambda, w) \in \mathbb{R}_+ \times X$; furthermore, the reverse $\overset{\leftarrow}{F}$ may be chosen such that $\overset{\leftarrow}{F}(\lambda * x) = \lambda \cdot \overset{\leftarrow}{F}(x)$.

<u>3.2.9 Remark</u>: By the above, if a transformation F on an input space X satisfies conditions (3.2.5-i, ii, iii) and has the further property: for all input $x \neq 0$, $\|F(\lambda \cdot x)\| \to +\infty$ as $\lambda \to +\infty$; then it is reasonable to call a non-$\{0\}$ equivalence class D_β, induced by F via (3.2.5.1), a *generalized ray* since it is the transformation image of a ray in the input space.

3.3 Stochastic Homothetic Production Correspondences

The notion of a homothetic production function introduced by Shephard [1953], and subsequently extended to production correspondences (see Shephard [1970-a], Färe/Shephard [1977]), has found wide application in economic theories. In this section, stochastic homothetic production correspondences are formulated in terms of scaling operations on the factor spaces.

As motivation, consider a mapping $F : X \times S \to X$ where S is the state space and X the input space of a production technology.

Suppose for each state $s \in S$, the mapping $x \in X \longmapsto F_s(x) := F(x,s)$ satisfies conditions (3.2.5-i, ii, iii). Then by Proposition (3.2.5), each of the mappings F_s induces a normal scaling operation, denoted $*_s$, on the input space. Given a deterministic input correspondence $\hat{L} : U \to 2(X)$ which is ray homothetic with a scaling law χ; i.e., $\hat{L}(\theta \cdot u) = \chi(\theta,u) \cdot \hat{L}(u)$, $\theta \in \mathbb{R}_{++}$, $u \in U$ (see Färe/Shephard [1977] and Eichhorn [1970] for a rigorous treatment of ray-homotheticity and generalized homogeneity); the correspondence defined by

$$\text{(3.3.1)} \quad (u,s) \in U \times S \longmapsto L(u,s) := \{x \in X \mid x = F(y,s) \ , \ y \in \hat{L}(u)\}$$

may be easily shown (using Proposition (3.2.8)) to satisfy:

$$\text{(3.3.2)} \qquad L(\theta \cdot u,s) = \chi(\theta,u) *_s L(u,s) \ , \ (\theta,u) \in \mathbb{R}_{++} \times U \ .$$

Motivated by the functional form of (3.3.2), one makes:

<u>(3.3.3) Definition</u>: A stochastic input correspondence $L : U \times S \to 2(X)$ has a *stochastic ray scale homothetic structure* if it satisfies a functional equation of the form

$$L(\theta \cdot u,s) = \chi(\theta,u,s) *_s L(u,s) \ , \quad (\theta,u,s) \in \mathbb{R}_{++} \times U \times S \ ;$$

where $\chi : \mathbb{R}_{++} \times U \times S \to \mathbb{R}_{++}$, $\chi(1,u,s) = 1 = \chi(\theta,0,s)$ for all $(\theta,u,s) \in \mathbb{R}_{++} \times U \times S$; and $*_s$ is a scaling operation on the input space X depending on the state $s \in S$.

For simplicity, the scaling operation on the output space U is taken to be radial in the above definition. It should be noted that uncertainty enters into a homothetic structure in three ways: (a) the scaling law χ; (b) the scaling operation $*_s$; and (c) the input

sets $L(u,s)$. Of course, when the state space S is a singleton, Definition (3.3.3) reduces to a generalization of the deterministic ray homothetic input correspondence (replacing the usual radial scaling by a scaling operation, see Mak [1980-a]).

In order for the input correspondence L defined above to be a model of a stochastic technology, it is assumed to satisfy {L0, L1, L2, L3, L4.1, L4.2, L5 and L6} as stated with scaling operations $*_s$, $s \in S$. Henceforth, *this assumption will be imposed* in the exposition of this chapter.

Clearly, (3.3.1) as generated by the transformation $F : X \times S \to X$ is a special case of Definition (3.3.3). In fact, (3.3.3) is quite a general model of technology since the scaling operation $*_s$ and the input sets $L(u,s)$ may take on rather different forms. However, it is exceedingly difficult to study concretely a technology if there are no explicit relationships between the scaling operations $*_s$, or the input sets $L(u,s)$ as the state s varies. Hence, it is useful to postulate further special structures on the technology. The following two structures are prototypes of the others to come.

<u>(3.3.4) Definition</u>: A stochastic input correspondence $L : U \times S \to 2^{(X)}$ has an *invariant scaling structure* (IS for short) if it is stochastic ray scale homothetic with both the scaling law χ and the scaling operation on inputs independent of the state of production environment; explicitly, for some scaling operation $*$ on X , $L(\theta \cdot u,s) = \chi(\theta,u) * L(u,s)$ for all $(\theta,u,s) \in \mathbb{R}_{++} \times U \times S$.

With respect to a given scaling operation $*$ on an input space X, two sets Y, Z in X are said to be of the *same shape* if there exists a $\mu \in \mathbb{R}_{++}$ such that $Y = \mu * Z$.

<u>(3.3.5) Definition</u>: A stochastic input correspondence $L : U \times S \to 2(X)$ has an *invariant shape structure* (SS for short, signifying "same shapedness") if it is ray scale homothetic with a scaling operation $*$ on X which is independent of the state; and for all $(u,s,\bar{s}) \in U \times S \times S$, $L(u,s)$ and $L(u,\bar{s})$ are of the same shape (with respect to $*$).

Given a deterministic ray homothetic input correspondence $\hat{L} : U \to 2(X)$ and a scaling operation $*$ on the input space X, an IS structured stochastic input correspondence L may be generated by the following procedure:

(3.3.6.1) Mapping $J : X \times S \to X$; for each $s \in S$, $J(\cdot,s)$ is invertible and $J(\mu \cdot x,s) = \mu * J(x,s)$ for all $(\mu,x,s) \in \mathbb{R}_{++} \times X \times S$.

(3.3.6.2) $(u,s) \in U \times S \to L(u,s) := \{x \in X \mid x = J(y,s)\ ,\ y \in \hat{L}(u)\}$.

Similarly, an SS structured stochastic input correspondence L may be generated as follows:

(3.3.6.3) Mapping $M : X \times U \times S \to X$ is separable in the sense that for some $W : U \times S \to \mathbb{R}_{++}$, $M(y,u,s) = W(u,s)*F(y)$ for all $(y,u,s) \in X \times U \times S$ where the mapping F induces $*$ on X.

(3.3.6.4) $(u,s) \in U \times S \longmapsto L(u,s) := \{x \in X \mid x = M(y,u,s)\ ,\ y \in \hat{L}(u)\}$.

If the deterministic input correspondence $\hat{L}$ has a scaling law χ, it follows immediately from (3.3.6.3) and (3.3.6.4) that

$$L(\theta\cdot u,s) = \left[\frac{W(\theta\cdot u,s)}{W(u,s)}\cdot \chi(\theta,u)\right]*L(u,s) \quad , \quad (\theta,u,s) \in \mathbb{R}_{++} \times U \times S \ ;$$

and

$$L(u,s) = \frac{W(u,s)}{W(u,\bar{s})} * L(u,\bar{s}) \quad , \quad (u,s,\bar{s}) \in U \times S \times S \ .$$

Both the IS and SS structure has an immediate implication on their associated confidence index production correspondence (see (1.7.3) and (1.7.10) for definition) which is stated after the following:

<u>(3.3.7) Definition</u>: A scaling operation $*$ on an input space X is *continuous* if for every sequence $\{x^k\} \subset X$ converging to x^o, and every sequence $\{\mu^k\} \subset \mathbb{R}_{++}$ converging to $\mu^o \in \mathbb{R}_+$, $\{\mu^k * x^k\}$ converges to $\mu^o * x^o$.

<u>(3.3.8) Proposition</u>: Consider a stochastic input correspondence $L : U \times S \to 2(X)$. If L has an IS structure, then its confidence indexed input correspondence $CL : U \times [0,1] \to 2(X)$ is ray scale homothetic: for every $(u,\xi) \in U \times [0,1]$ and $\theta \in \mathbb{R}_{++}$, $CL(\theta\cdot u,\xi) = \chi(\theta,u)*CL(u,\xi)$ for some scaling law $\chi : \mathbb{R}_{++} \times U \to \mathbb{R}_{++}$.
If L has a SS structure and (a) the scaling operation $*$ is continuous; (b) the associated CL correspondence satisfies property CL5, see Proposition (1.7.5); then for each $u \in U$, the sets $CL(\theta\cdot u,\xi)$ which are not empty have the same shape as (θ,ξ) varies over $\mathbb{R}_{++} \times [0,1]$; implying that the correspondence CL is ray scale homothetic.

<u>Proof</u>: Suppose $L : U \times S \to 2(X)$ has an IS structure. Arbitrarily fix an $(\bar{u},\bar{\xi}) \in U \times [0,1]$; $CL(\bar{u},\bar{\xi})$ may be represented as

$$CL(\bar{u},\bar{\xi}) = \{x \in X \mid \{s \in S \mid x \in L(\bar{u},s)\} \succeq \bar{\xi}\}$$

$$= \bigcup_{A \in \mathcal{A}(\bar{\xi})} \bigcap_{s \in A} L(\bar{u},s)$$

where $\mathcal{A}(\bar{\xi})$ is the collection of events $\{A \in \mathcal{A} \mid \mathcal{P}(A) \succeq \bar{\xi}\}$. Since L has an IS structure with an invariant scaling law, say χ , it follows that for all $\theta \in \mathbb{R}_{++}$,

$$CL(\theta \cdot \bar{u},\bar{\xi}) = \bigcup_{A \in \mathcal{A}(\bar{\xi})} \bigcap_{s \in A} L(\theta\bar{u},s) = \bigcup_{A \in \mathcal{A}(\bar{\xi})} \bigcap_{s \in A} \chi(\theta,\bar{u}) * L(\bar{u},s)$$

$$= \chi(\theta,\bar{u}) * CL(\bar{u},\bar{\xi}) \ .$$

That is, CL is ray scale homothetic.

Suppose $L : U \times S \to 2(X)$ has a SS structure. First note that $CL(u,\xi = 0) \equiv X$ for all $u \in U$ and $CL(u = 0,\xi) \equiv X$ for all $\xi \in [0,1]$; hence, only the case of $\xi \in (0,1]$ and $u \neq 0$ need to be considered. Fix an arbitrary $\bar{u} \in U$, $\bar{u} \neq 0$. If $L(\bar{u},s) = \emptyset$ for all $s \in S$, then clearly $CL(\bar{u},\xi) = \emptyset$ for all $\xi \in (0,1]$. Furthermore, by Axiom L4.2, $L(\theta \cdot \bar{u},s)$ is seen to be empty for all $\theta \in \mathbb{R}_{++}$. Hence the proposition is trivially true. So, suppose for some $\bar{s} \in S$, $L(\bar{u},\bar{s}) \neq \emptyset$. Fix the input set $L(\bar{u},\bar{s})$ as a reference set and denote it simply as D . Since L has an SS structure, for each $s \in S$, there is an $\lambda_s \in \mathbb{R}_{++}$ such that $L(\bar{u},s) = \lambda_s * D$.

Consider an arbitrary event $A \in \mathcal{A}$ with $\bigcap_{s \in A} L(\bar{u},s) \neq \emptyset$. According to the above consideration, $\bigcap_{s \in A} L(\bar{u},s) = \bigcap_{s \in A} \lambda_s * D$. By L3 as stated with the scaling operation $*$, it is clearly true that $\left(\operatorname{Sup}_{s \in A} \lambda_s\right) * D \subset \bigcap_{s \in A} \lambda_s * D$. To show the converse inclusion, let $\bar{x} \in \bigcap_{s \in A} \lambda_s * D$. Let $\sigma := \operatorname{Min} \{\lambda \in \mathbb{R}_{++} \mid \lambda * \bar{x} \in D\}$. The scalar σ is well-defined since

$D \equiv L(\bar{u},\bar{s})$ is closed (by Axiom L.5) and $*$ is a continuous scaling operation. Furthermore, $\sigma > 0$; since if otherwise, $\sigma * \bar{x} = 0 \in L(\bar{u},\bar{s})$, contradicting L1. By Axiom L3 and the definition of σ , $1/\sigma \geqq \lambda_s$ for all $s \in A$; implying that $1/\sigma \geqq \operatorname{Sup}_{s \in A} \lambda_s$. Since $\sigma * \bar{x} \in D$, $\bar{x} \in 1/\sigma * D$. Then by L3 as applied on $D \equiv L(\bar{u},\bar{s})$, $\bar{x} \in \left(\operatorname{Sup}_{s \in A} \lambda_s\right) * D$.

Let $\bar{\xi}$ be an arbitrary confidence index in $(0,1]$. Let $\mathcal{A}$ be the family of events in $\mathcal{S}$ defined by : - $A \in \mathcal{A}$ iff $\mathcal{P}(A) \geqq \bar{\xi}$ and $\bigcap_{s \in A} \lambda_s * D \neq \emptyset$. Suppose $CL(\bar{u},\bar{\xi})$ is not empty. Then $\mathcal{A}$ is a non-null family. For each event $A \in \mathcal{A}$, let $\lambda^A := \operatorname{Sup}_{s \in A} \lambda_s$. Then $CL(\bar{u},\bar{\xi})$ has the representation of $\bigcup_{A \in \mathcal{A}} [\lambda^A * D]$. From this, it is seen that $CL(\bar{u},\bar{\xi})$ and D are of the same shape if one can show that $\bigcup_{A \in \mathcal{A}} [\lambda^A * D] = \left(\operatorname{Inf}_{A \in \mathcal{A}} \lambda^A\right) * D$ and $\left(\operatorname{Inf}_{A \in \mathcal{A}} \lambda^A\right) \in \mathbb{R}_{++}$.

Since $\bar{u} \neq 0$ and $\bar{\xi} > 0$, $0 \notin CL(\bar{u},\bar{\xi})$. Since $CL(\bar{u},\bar{\xi})$ is assumed to be closed, there is a neighborhood N of 0 such that $N \cap CL(\bar{u},\bar{\xi}) = \emptyset$. Consider an arbitrary $x' \in D$. By the continuity of the scaling operation $*$, there is a $\mu' \in \mathbb{R}_{++}$ so small that $\mu' * x' \in N$. Denote $\mu' * x'$ by z . Clearly $z \notin CL(\bar{u},\bar{\mu})$, $z \neq 0$ and the generalized ray $\{\lambda * z \mid \lambda \in \mathbb{R}_{++}\}$ has a nonempty intersection with D . Let $\beta := \operatorname{Min} \{\lambda \in \mathbb{R}_{++} \mid \lambda * z \in D\}$. As argued before, β is well-defined and positive. Since z does not belong to the closed set $CL(\bar{u},\bar{\xi})$, Axiom L3 and the continuity of $*$ implies the existence of a $\delta \in \mathbb{R}_{++}$ such that for all $\mu \in [0,1+\delta)$, $\mu * z \notin CL(\bar{u},\bar{\xi}) \equiv \bigcup_{A \in \mathcal{A}} [\lambda^A * D]$. Clearly $\lambda^A \in \mathbb{R}_{++}$ for all $A \in \mathcal{A}$. Since $\beta * z \in D$, $(\beta\lambda^A) * z \in \lambda^A * D$ for all $A \in \mathcal{A}$. Consequently, $\beta\lambda^A \geqq 1 + \delta$ for all A ; implying that

$$\left(\operatorname{Inf}_{A \in \mathcal{A}} \lambda^A\right) \geqq \frac{1+\delta}{\beta} > 0 .$$

For ease of notation, denote $\operatorname{Inf}_{A\in\mathcal{A}} \lambda^A$ simply by $\operatorname{Inf} \lambda^A$. Since $\operatorname{Inf} \lambda^A$ was shown to be positive, it follows from L3 that $\bigcup_{A\in\mathcal{A}} [\lambda^A * D]$ is contained in $(\operatorname{Inf} \lambda^A)*D$. To show the converse inclusion, let $\bar{z} \in (\operatorname{Inf} \lambda^A)*D$. Then the generalized ray $\{\lambda*\bar{z} \mid \lambda \in \mathbb{R}_{++}\}$ clearly intersects D. To use contra-positive argument, suppose $\bar{z} \notin CL(\bar{u},\bar{\xi})$. Then by defining $\gamma := \operatorname{Min} \{\lambda \in \mathbb{R}_{++} \mid \lambda*\bar{z} \in D\}$, it may be argued as in the above paragraph that there exists $\varepsilon \in \mathbb{R}_{++}$ such that $\operatorname{Inf} \lambda^A \geqq \frac{1+\varepsilon}{\gamma}$. Now $\bar{z} \in (\operatorname{Inf} \lambda^A)*D$ implies $(1/\operatorname{Inf} \lambda^A)*\bar{z} \in D$. Hence $1/\operatorname{Inf} \lambda^A \geqq \gamma$ by the definition of γ. But this contradicts the inequality $\operatorname{Inf} \lambda^A \geqq \frac{1+\varepsilon}{\gamma}$ since $\varepsilon > 0$.

By the above argument, it has been shown that for each $\xi \in (0,1]$ with $CL(\bar{u},\xi) \neq \emptyset$, $CL(\bar{u},\xi)$ and $D\ (\equiv L(\bar{u},\bar{s}))$ are of the same shape. Now by the ray scale homotheticity of L, for every $\theta \in \mathbb{R}_{++}$, $L(\theta\cdot\bar{u},\bar{s})$ is of the same shape as $L(\bar{u},\bar{s})$. Using $L(\theta\cdot\bar{u},\bar{s})$ as the reference set and repeating the argument above, it is seen that for all $\theta \in \mathbb{R}_{++}$, $\xi \in (0,1]$, $CL(\theta\cdot\bar{u},\xi)$ is of the same shape as $L(\theta\cdot\bar{u},\bar{s})$, hence that of D $\square$

The next two representation propositions will further expose the structure of IS and SS stochastic input correspondences. Before stating them, it is recalled that a production function $\Phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is said to be homothetic if it has the form (see Shephard [1970-a]):

$$\Phi(x) = G(\phi(x))$$

where $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ is homogenous and $G : \mathbb{R}_+ \to \mathbb{R}_+$ has the properties

$$(3.3.9)\quad \begin{array}{rl} \text{(i)} & G(0) = 0\ ;\ G \text{ is nondecreasing;} \\ \text{(ii)} & G(\alpha) \to +\infty \text{ as } \alpha \to +\infty\ ; \\ \text{(iii)} & G \text{ is upper-semi-continuous; and} \\ \text{(iv)} & \beta \in \mathbb{R}_+ \mapsto G^{-1}(\beta) := \text{Min}\,\{\alpha \in \mathbb{R}_+ \mid G(\alpha) \geqq \beta\}\ . \end{array}$$

<u>(3.3.10) Proposition</u>: Suppose a stochastic input correspondence $L : U \times S \to 2(X)$ has a SS structure with a scaling operation $*$ (on X) which is continuous. Then for each output mix $\frac{u}{\|u\|} \in \Gamma U := \{w \mid \|w\| = 1\}$, there is a family of functions $G_s\left(\cdot, \frac{u}{\|u\|}\right) : \mathbb{R}_+ \to \mathbb{R}_+$; $s \in S$, satisfying properties (3.3.9-i, ii, iii) and a *scale homogenous* function $\phi\left(\cdot, \frac{u}{\|u\|}\right) : X \to \mathbb{R}_+$ $\left(\text{i.e., } \phi\left(\lambda x, \frac{u}{\|u\|}\right) = \lambda * \phi\left(x, \frac{u}{\|u\|}\right)\right)$ such that $L\left(\theta \cdot \frac{u}{\|u\|}, s\right) = \left\{x \in X \mid G_s\left(\phi\left(x, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right) \geqq \theta\right\}$ for all $(\theta, s) \in \mathbb{R}_{++} \times S$.

<u>*Proof*</u>: *Arbitrarily fix an output mix* $\frac{u}{\|u\|} \in \Gamma U$. Suppose $L\left(\frac{u}{\|u\|}, s\right) = \emptyset$ for all $s \in S$. Define $\phi\left(x, \frac{u}{\|u\|}\right) = 0$ for all $x \in X$. Clearly, $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ thus defined is scale homogenous. For all $s \in S$, let $G_s\left(\cdot, \frac{u}{\|u\|}\right)$ be an arbitrary function satisfying (3.3.9-i, ii, iii). Then it is seen that the representation of the input sets $L\left(\theta \cdot \frac{u}{\|u\|}, s\right)$ as claimed by the proposition is valid for all $\theta \in \mathbb{R}_{++}$.

Suppose $L\left(\frac{u}{\|u\|}, s\right) \neq \emptyset$ for some $s \in S$. Then since L is SS structured, $L\left(\frac{u}{\|u\|}, s\right) \neq \emptyset$ for all $s \in S$. Then as argued in the proof of Proposition (3.3.8), there is a closed subset D in X $(0 \notin D)$ such that $L\left(\theta \cdot \frac{u}{\|u\|}, s\right)$ is of the same shape as D for all $(\theta, s) \in \mathbb{R}_{++} \times S$. That is, for each $(\theta, s) \in \mathbb{R}_{++} \times S$, there is a $q(\theta, s) \in \mathbb{R}_{++}$ with $L\left(\theta \cdot \frac{u}{\|u\|}, s\right) = q(\theta, s) * D$. D of course depends on $\frac{u}{\|u\|}$.

Now fix a state $\bar{s} \in S$. It will be first shown that the function $\theta \in \mathbb{R}_{++} \longmapsto q(\theta, \bar{s})$ is lower-semi-continuous. Consider an arbitrary

sequence of scalars $\{\theta^k\} \subset \mathbb{R}_{++}$ converging to $\theta^o \in \mathbb{R}_{++}$. Let a subsequence $\{\theta^j\} \subset \{\theta^k\}$ has the property that $\{q(\theta^j,\bar{s})\}$ converges to $q := \liminf \{q(\theta^k,\bar{s})\}$. Consider an arbitrary $\bar{x} \in D$. By the definition of the function $q(\cdot,\bar{s})$, $q(\theta^j,\bar{s})*\bar{x} \in L\left(\theta^j \cdot \frac{u}{\|u\|},\bar{s}\right)$ for each index j. Since $\{q(\theta^j,\bar{s})\}$ converges to q, and the scaling operation $*$ is continuous, $\{q(\theta^j,\bar{s})*\bar{x}\}$ converges to $q*\bar{x}$. Then since $\left\{\theta^j \cdot \frac{u}{\|u\|}\right\}$ converges to $\theta^o \cdot \frac{u}{\|u\|}$, by Axiom L5 $q*\bar{x} \in L\left(\theta^o \cdot \frac{u}{\|u\|},\bar{s}\right)$. Because $\bar{x}$ was arbitrarily chosen, $q*D \subset L\left(\theta^o \cdot \frac{u}{\|u\|},\bar{s}\right)$. But because $L\left(\theta^o \cdot \frac{u}{\|u\|},\bar{s}\right) = q(\theta^o,\bar{s})*D$, it follows (by applying L3) that $q(\theta^o,\bar{s}) \leqq q = \liminf \{q(\theta^k,\bar{s})\}$.

Moreover, it is not difficult to show (using L3 and L2 respectively) that the function $q(\cdot,\bar{s})$ is nondecreasing and $q(\theta,\bar{s}) \to +\infty$ as $\theta \to +\infty$. Using these properties of $q(\cdot,\bar{s})$, define a function $G_{\bar{s}}\left(\cdot, \frac{u}{\|u\|}\right)$: $\mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\alpha \in \mathbb{R}_+ \to G_s\left(\alpha, \frac{u}{\|u\|}\right) = \begin{cases} 0, \text{ if } \{\theta \in \mathbb{R}_{++} \mid q(\theta,\bar{s}) \leqq \alpha\} = \emptyset ; \\ \text{Max } \{\theta \in \mathbb{R}_{++} \mid q(\theta,\bar{s}) \leqq \alpha\} \text{ if otherwise.} \end{cases}$$

The function $G_{\bar{s}}\left(\cdot, \frac{u}{\|u\|}\right)$ is well defined; nonnegative; $G(0) = 0$; upper semi-continuous and $G_{\bar{s}}\left(\alpha, \frac{u}{\|u\|}\right) \to +\infty$ as $\alpha \to +\infty$. (See Shephard [1970-a, Proposition 6] for details.) That is, $G_{\bar{s}}\left(\cdot, \frac{u}{\|u\|}\right)$ satisfies (3.3.9-i, ii, iii). Furthermore,

$$(3.3.10.1) \quad \alpha \geqq q(\theta,\bar{s}) \text{ iff } G_{\bar{s}}\left(\alpha, \frac{u}{\|u\|}\right) \geqq \theta \text{ , all } (\theta,\alpha) \in \mathbb{R}_{++} \times \mathbb{R}_+ .$$

Repeating the above argument for each $s \in S$, a family of functions $\left\{G_s\left(\cdot, \frac{u}{\|u\|}\right); s \in S\right\}$ satisfying (3.3.9-i, ii, iii) is defined.

Next, define a function $\phi\left(\cdot, \frac{u}{\|u\|}\right): X \to \mathbb{R}_+$ as follows:

$$\phi\left(x, \frac{u}{\|u\|}\right) := \begin{cases} 0 \ , \text{ if } \ \{\lambda * x \mid \lambda \in \mathbb{R}_{++}\} \cap D = \emptyset \ ; \\ [\operatorname{Min} \{\lambda \in \mathbb{R}_{++} \mid \lambda * x \in D\}]^{-1} \ , \text{ if otherwise.} \end{cases}$$

The function $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ is well defined since D is closed, $0 \notin D$ and $*$ is continuous; as argued once in the previous proposition. In fact, it is scale-analogue of the distance function of the input set D (see Shephard [1970-a] and Shephard/Färe [1980] for a definition of distance function and its properties). Clearly, $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ is scale homogenous. Moreover, $D \equiv \left\{x \in X \mid \phi\left(x, \frac{u}{\|u\|}\right) \geqq 1\right\}$. For a proof of this fact when $*$ is the usual radial scaling, see Shephard [1970-a, Proposition 16].

Finally, using (3.3.10.1) and the scale homogeneity of $\phi\left(\cdot, \frac{u}{\|u\|}\right)$, one has: for all $(\theta, s) \in \mathbb{R}_{++} \times S$

$$\begin{aligned} L\left(\theta \cdot \frac{u}{\|u\|}, s\right) &= \{x \in X \mid x = q(\theta,s) * y \ , \ y \in D\} \\ &= \left\{x \in X \mid x = q(\theta,s) * y \ , \ \phi\left(y, \frac{u}{\|u\|}\right) \geqq 1\right\} \\ &= \left\{z \in X \mid \phi\left(z, \frac{u}{\|u\|}\right) \geqq q(\theta,s)\right\} \\ &= \left\{z \in X \mid G_s\left(\phi\left(z, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right) \geqq \theta\right\} . \end{aligned}$$

Since the mix $\frac{u}{\|u\|}$ was arbitrarily chosen, the proposition is established □

<u>(3.3.11) Proposition</u>: Suppose a stochastic input correspondence $L : U \times S \to 2(X)$ has an IS structure with a continuous scaling operation $*$ and a scaling law $\chi : \mathbb{R}_{++} \times U \to \mathbb{R}_{++}$. Then for each output mix $\frac{u}{\|u\|} \in \Gamma U$, there exists a function $G\left(\cdot, \frac{u}{\|u\|}\right): \mathbb{R}_+ \to \mathbb{R}_+$

satisfying (3.3.9-i, ii, iii) and a family of scale homogenous functions $\left\{\phi_s\left(\cdot, \frac{u}{\|u\|}\right) : X \to \mathbb{R}_+ \; ; \; s \in S\right\}$ such that $L\left(\theta \cdot \frac{u}{\|u\|}, s\right) = \left\{x \in X \mid G\left(\phi_s\left(x, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right) \geqq \theta\right\}$ for all $(\theta, s) \in \mathbb{R}_{++} \times S$.

<u>Proof</u>: Similar to that of (3.3.10) and will be omitted.

Based on the representation propositions (3.3.10) and (3.3.11), one may choose to interpret the SS and IS structures as constituted of scalar-valued production functions $\left(G_s\left(\cdot, \frac{u}{\|u\|}\right)\right.$ and $\left.G\left(\cdot, \frac{u}{\|u\|}\right)\right)$ and input quantity indices $\left(\phi\left(\cdot, \frac{u}{\|u\|}\right)\right.$ and $\left.\phi_s\left(\cdot, \frac{u}{\|u\|}\right)\right)$. However, as the proof of Proposition (3.3.10) reflects, the representations are not unique (since the reference set D was chosen arbitrarily). Hence, such an interpretation may be somewhat strained. But these representations motivate the following considerations:

As in (1.7.9) and (2.2.10), define for each $\frac{u}{\|u\|} \in \Gamma U$ the function

$$\text{(3.3.12)} \quad (x,s) \in X \times S \longmapsto \Phi\left(x, s \mid \frac{u}{\|u\|}\right) := \text{Max}\left\{\sigma \in \mathbb{R}_+ \mid \sigma \cdot \frac{u}{\|u\|} \in P(x,s)\right\}.$$

As argued before, it follows from Axioms L5 and P0 that Φ is well defined, and for every $x \in X$, the function $\Phi\left(x, \cdot \mid \frac{u}{\|u\|}\right)$ is measurable. Hence $\left\{\Phi\left(x, \cdot \mid \frac{u}{\|u\|}\right) ; x \in X\right\}$ may be taken as the family of random variables which models the production function for outputs of mix $\frac{u}{\|u\|}$.

Suppose the input correspondence L of a stochastic production technology has a SS structure. Let the *effective domain* of output mix be $DU := \left\{\frac{u}{\|u\|} \in \Gamma U \mid L\left(\frac{u}{\|u\|}, s\right) \neq \emptyset \text{ for some } s \in S\right\}$. Suppose L is represented by the families $\left\{G_s\left(\cdot, \frac{u}{\|u\|}\right) ; s \in S\right\}$ and scale homogenous functions $\phi\left(\cdot, \frac{u}{\|u\|}\right)$, $\frac{u}{\|u\|} \in \Gamma U$. Then, as may be seen from the proof of

(3.3.10), for each $\frac{u}{\|u\|} \in DU$, $\phi\left(x, \frac{u}{\|u\|}\right)$ takes on all possible values of $\mathbb{R}_+$ as x varies over X . Consider an arbitrary $\bar{\alpha} \in \mathbb{R}_+$, it is easy to verify that for every $x \in X$ with $\phi\left(x, \frac{u}{\|u\|}\right) = \bar{\alpha}$ and $\frac{u}{\|u\|} \in DU$: $-\left\{s \in S \mid G_s\left(\bar{\alpha}, \frac{u}{\|u\|}\right) \geqq \sigma\right\} = \left\{s \in S \mid \Phi\left(x,s \mid \frac{u}{\|u\|}\right) \geqq \sigma\right\} \in \mathcal{B}$, $\sigma \in \mathbb{R}_+$.

Thus, for each $\frac{u}{\|u\|} \in DU$, the family of random variables $\left\{\Phi\left(x,\cdot \mid \frac{u}{\|u\|}\right);\ x \in X\right\}$ is representable by a single function $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ and the family $\left\{G_s\left(\cdot, \frac{u}{\|u\|}\right),\ s \in S\right\}$. Moreover, for a fixed $\alpha \in \mathbb{R}_+$ and a $\frac{u}{\|u\|} \in DU$, the function $s \in S \longmapsto G_s\left(\alpha, \frac{u}{\|u\|}\right)$ is a random variable with a distribution

$$(3.3.13)\quad V\left(\theta \mid \alpha, \frac{u}{\|u\|}\right) := \mathcal{P}\left\{s \in S \mid G_s\left(\alpha, \frac{u}{\|u\|}\right) \leqq \theta\right\},\ \theta \in \mathbb{R}_+ .$$

In general, for a fixed $\frac{u}{\|u\|} \in DU$, the distribution function $V\left(\cdot \mid \alpha, \frac{u}{\|u\|}\right)$ are different for different α's . However, if these distributions are of manageable complexity, the discussion above is potentially useful for application. For example:

<u>(3.3.14) Definition</u>: A SS structured stochastic input correspondence $L : U \times S \to 2(X)$ (with a continuous scaling operati-- *) is said to have a SSG *structure* if its representation (3.3.10) satisfies: given $\frac{u}{\|u\|} \in DU$, for every s , $\bar{s} \in S$, there exists a scalar $\beta\left(s,\bar{s}, \frac{u}{\|u\|}\right)$ such that $G_s\left(\alpha, \frac{u}{\|u\|}\right) = \beta\left(s,\bar{s}, \frac{u}{\|u\|}\right) \cdot G_{\bar{s}}\left(\alpha, \frac{u}{\|u\|}\right)$, $\alpha \in \mathbb{R}_+$.

<u>(3.3.15) Proposition</u>: Suppose a stochastic input correspondence L has a SSG structure. Then the distribution functions V (3.3.13) associated with its representation satisfies: for every $\frac{u}{\|u\|} \in DU$ and every α' , $\alpha'' \in \{\alpha \in \mathbb{R}_+ \mid G_s(\alpha) > 0$ for some $s \in S\}$, there exists

a scalar $g\left(\alpha',\alpha'', \frac{u}{\|u\|}\right) \in \mathbb{R}_{++}$ such that $V\left(t \mid \alpha', \frac{u}{\|u\|}\right) =$ $V\left(g\left(\alpha',\alpha'', \frac{u}{\|u\|}\right) \cdot t \mid \alpha'', \frac{u}{\|u\|}\right)$ for all $t \in \mathbb{R}_+$. In fact, $g\left(\alpha',\alpha'', \frac{u}{\|u\|}\right) \equiv G_s\left(\alpha', \frac{u}{\|u\|}\right) \Big/ G_s\left(\alpha'', \frac{u}{\|u\|}\right)$ for each $s \in S$. From this, it follows that if the distribution functions V are integrable, denoting $\mathrm{EXP}\left[\phi\left(x \mid \frac{u}{\|u\|}\right)\right] := \int_{s \in S} \phi\left(x,s \mid \frac{u}{\|u\|}\right) d\mathcal{P}$, one has

$$\frac{\mathrm{EXP}\left[\phi\left(x \mid \frac{u}{\|u\|}\right)\right]}{\mathrm{EXP}\left[\phi\left(y \mid \frac{u}{\|u\|}\right)\right]} = \frac{G_s\left(\phi\left(x, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right)}{G_s\left(\phi\left(y, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right)}, \text{ all } x, y \in X \text{ and } s \in S .$$

<u>Proof</u>: Straightforward and omitted.

Although the data requirement of working with a SSG structure is minimal $\Big($a single scale homogenous function $\phi\left(\cdot, \frac{u}{\|u\|}\right)$; a single scalar-valued production $G_s\left(\cdot, \frac{u}{\|u\|}\right)$, s being an arbitrary element of S ; and a single distribution function $V\Big)$; SSG structures appear to be too simplistic. But based on it, a rather useful extension is given below:

<u>(3.3.16) Definition</u>: A stochastic input correspondence $L : U \times S \to 2(X)$ has a *partially-same-shape structure* (PSS for short) if the state space S is partitioned by $\{S^j , j \in J\}$; and for each member S^j of the partition, the restricted input correspondence $L^j : U \times S^j \to 2(X)$ defined by $L^j(u,s) := L(u,s)$ $(u \in U , s \in S^j)$ has a SS structure. Input correspondence L is of PSSG structure if each L^j is of SSG structure.

It is assumed that L^j's satisfies the stochastic weak axioms (1.4.6) as stated with appropriate scaling operations which could be different for different indices. The relevant state space $(S^j, \mathcal{S}^j)$ is the restriction of $(S,\mathcal{S})$ on S^j .

The rationale underlying the PSS structure is the observation that in many production systems, the state of the production environment affects production in *distinctively* and *qualitatively* different manners. For instance, whether it rains or not makes a tremendous difference to many construction projects; but given that it rains, the amount of rainfall is irrelevant. Another example is technological breakthrough. After a breakthrough which qualitatively changes the technique of production, only minor variation is effected by the state of production. PSS structures attempt to capture this categorization of qualitative difference of the technology.

Recall the notion of an information structure as represented by a sequence of partition on the state space S. Consider the information at time t as represented by the partition $\mathcal{J}_t$. Suppose $\mathcal{J}_t$ is finer than the partition $\{S^j, j \in J\}$ of a PSS technology. Then the information at time t enables a producer to tell precisely which qualitatively distinct input correspondence is prevalent. This example indicates that PSS structures may be particularly useful in formulating production policies.

3.4 Homothetic Structures and Production Decisions

The last section introduces the notion of stochastic homothetic production structures, although primarily through some rather special forms. In this section, the special homothetic structures developed are used to consider some production planning and policy problems. Since the underlying technology is assumed to have rather special structures, the material in this section should only be regard as an exploration preliminary to the study of production planning under uncertainty.

The remainder of this chapter is divided into two parts. First, design of production systems is considered via the confidence indexed correspondences. Using the first part as background, in the next section, a simple stochastic dynamic production system is investigated with the constraint of information explicitly introduced.

By overall planning and design of a production system, it is meant that certain decisions concerning the input to and output from a production system are to be made at a particular time point without explicit concern for their execution (the day-to-day system operation under uncertainty). Examples are: planning for investment on plant capacity; production target setting, etc. Under this framework of decision-making, the information (or ignorance) of a producer is completely embodied in the state space $(S,\mathcal{B})$ and the (subjective) probability measure. Two schemes are considered:

(3.4.0-i) Output $\bar{u} \in U$ is to be attained with at least a confidence level $\bar{\xi}$; choose an input $x \in X$ which may accomplish this.

(3.4.0-ii) Input resources are constrained in some manner, choose a feasible input such that the expected output is optimal.

<u>(3.4.1) Definition</u>: Let X^* , the "dual" to an input space X , represent the space of nonnegative input prices. (Strictly speaking, this is an abuse of mathematical language. If $X = \mathbb{R}^n_+$, X^* is meant to be $\mathbb{R}^n_+$. Similarly, for $X = (L_\infty)^n_+$ with the weak* topology, X^* is meant to be $(L_1)^n_+$; for $X = (L_1)^n_+$, X^* is meant to be $(L_\infty)^n_+$.) The *(confidence indexed) minimal cost function* of a stochastic production technology is

$$(3.4.1.1)\quad (u,p,\xi) \in U \times X^* \times [0,1] \to Q(u,p,\xi) := \begin{cases} \operatorname{Inf}\{\langle p,x\rangle \mid x \in CL(u,\xi)\}, \\ \qquad \text{if} \quad CL(u,\xi) \neq \emptyset\ ; \\ +\infty\ , \text{ if otherwise.} \end{cases}$$

It is well known that the minimal cost function of a deterministic (ray) homothetic input correspondence is separable (see Shephard [1970-a] and Färe/Shephard [1977]). As an extension, it is shown in the following that certain classes of stochastic homothetic input correspondences also have separable (confidence indexed) minimal cost functions.

Suppose an input correspondence $L : U \times S \to 2(X)$ has an IS structure *with radial scaling on* X, then by Proposition (3.3.8), $CL(\theta \cdot u,\xi) = \chi(\theta,u) \cdot CL(u,\xi)$ for all $(\theta,u,\xi) \in \mathbb{R}_{++} \times U \times [0,1]$ where χ is the invariant scaling law. It then follows that for all $(\theta,u,p,\xi) \in \mathbb{R}_{++} \times U \times X^* \times [0,1]$

$$(3.4.2.1)\quad \begin{aligned} Q(\theta \cdot u,p,\xi) &= \operatorname{Inf}\{\langle p,x\rangle \mid x \in \chi(\theta,u) \cdot CL(u,\xi)\} \\ &= \chi(\theta,u) \cdot Q(u,p,\xi)\ . \end{aligned}$$

In particular, for a fixed output mix $\frac{u}{\|u\|} \in \Gamma U$,

$$(3.4.2.2)\quad Q\left(\theta \cdot \frac{u}{\|u\|}, p, \xi\right) = N_{\frac{u}{\|u\|}}(\theta) \cdot K_{\frac{u}{\|u\|}}(p,\xi)\ ,\ (\theta,p,\xi) \in \mathbb{R}_{++} \times X^* \times [0,1]\ ;$$

where $K_{\frac{u}{\|u\|}}(p,\xi) := Q\left(\frac{u}{\|u\|}, p, \xi\right)$ and $N_{\frac{u}{\|u\|}}(\theta) := \chi\left(\theta, \frac{u}{\|u\|}\right)$. That is, for each output mix $\frac{u}{\|u\|}$, the minimal cost Q separates into two terms, one involving the scaling of the output mix $\frac{u}{\|u\|}$, the other involving prices and the confidence index as applies to a reference output $\frac{u}{\|u\|}$.

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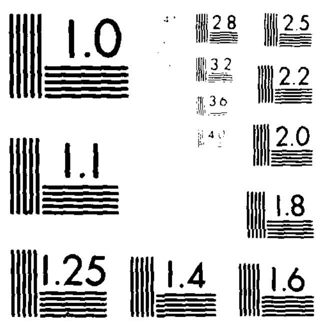

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Suppose $L : U \times S \to 2(X)$ has a SS structure with the usual radial scaling, then the minimal cost function Q is separable in quite a different manner. For simplicity, assume Property CP4.2 (hence CL4.2) holds. Let $\Xi(u) := \{\xi \in [0,1] \mid CL(u,\xi) \neq \emptyset\}$, $u \in U$. Noting that the radial scaling operation is continuous, it follows from Proposition (3.3.8) that for all $u \in U$ and ξ , $\xi' \in \Xi(u)$:

(3.4.3.1) for every $\theta \in \mathbb{R}_{++}$, there exists a scalar $\rho(\theta,u,\xi) \in \mathbb{R}_{++}$ such that $CL(\theta \cdot u,\xi) = \rho(\theta,u,\xi) \cdot CL(u,\xi)$;

(3.4.3.2) for some $\gamma(u,\xi,\xi') \in \mathbb{R}_{++}$, $CL(u,\xi) = \gamma(u,\xi,\xi') \cdot CL(u,\xi')$.

From (3.4.3.1), it directly follows that

(3.4.4.1) $Q(\theta \cdot u,p,\xi) = \rho(\theta,u,\xi) \cdot Q(u,p,\xi)$; $u \in U$, $\xi \in \Xi(u)$, $\theta \in \mathbb{R}_{++}$, $p \in X^*$.

In particular, for a fixed output mix $\frac{u}{\|u\|} \in \Gamma U$, by letting $K_{\frac{u}{\|u\|}}(p,\xi) := Q\left(\frac{u}{\|u\|}, p, \xi\right)$ and $M_{\frac{u}{\|u\|}}(\theta,\xi) := \rho\left(\theta, \frac{u}{\|u\|}, \xi\right)$, Equation (3.4.4.1) has the separable form:

(3.4.4.2) $$Q\left(\theta \cdot \frac{u}{\|u\|}, p, \xi\right) = M_{\frac{u}{\|u\|}}(\theta,\xi) \cdot K_{\frac{u}{\|u\|}}(p,\xi) \ .$$

On the other hand, it follows from (3.4.3.2) that

(3.4.5.1) $Q(u,p,\xi) = \gamma(u,\xi,\xi') \cdot Q(u,p,\xi')$; $u \in U$, ξ & $\xi' \in \Xi(u)$, $p \in X^*$.

In particular, for a fixed $\frac{u}{\|u\|} \in \Gamma U$ and an arbitrarily fixed $\xi' \in \Xi\left(\frac{u}{\|u\|}\right)$, Equation (3.4.5.1) gives rise to: - for all $p \in X^*$, $\xi \in \Xi\left(\frac{u}{\|u\|}\right)$:

$$Q\left(\theta \cdot \frac{u}{\|u\|}, p, \xi\right) = \gamma\left(\theta \cdot \frac{u}{\|u\|}, \xi, \xi'\right) \cdot Q\left(\theta \cdot \frac{u}{\|u\|}, p, \xi'\right)$$

$$(3.4.5.2) \qquad = \gamma\left(\theta \cdot \frac{u}{\|u\|}, \xi, \xi'\right) \cdot M_{\frac{u}{\|u\|}}(\theta, \xi') \cdot K_{\frac{u}{\|u\|}}(p, \xi')$$

$$= \Gamma_{\frac{u}{\|u\|}}(\theta, \xi) \cdot \tilde{M}_{\frac{u}{\|u\|}}(\theta) \cdot \tilde{K}_{\frac{u}{\|u\|}}(p)$$

with the obvious definitions for $\Gamma_{\frac{u}{\|u\|}}$, $\tilde{M}_{\frac{u}{\|u\|}}$ and $\tilde{K}_{\frac{u}{\|u\|}}$. Furthermore, if L has an IS structure in addition to being SS structured, it is easy to see that the factor $\Gamma_{\frac{u}{\|u\|}}(\theta, \xi)$ is really independent of θ, resulting in the completely separable form:

$$(3.4.6) \quad Q\left(\theta \cdot \frac{u}{\|u\|}, p, \xi\right) = \tilde{\Gamma}_{\frac{u}{\|u\|}}(\xi) \cdot \tilde{M}_{\frac{u}{\|u\|}}(\theta) \cdot \tilde{K}_{\frac{u}{\|u\|}}(p); \; p \in X^*, \; \theta \in \mathbb{R}_{++}, \; \xi \in \Xi\left(\frac{u}{\|u\|}\right).$$

The relevance of the above separable functional forms in regard to overall production planning (recall (3.5.0-i)) is as follows: - If the input correspondence L of a stochastic technology has the IS (or SS) structure with a radial scaling; and the value of $K_{\frac{u}{\|u\|}}(p, \xi)$ $\left(\text{or } K_{\frac{u}{\|u\|}}(p)\right)$ in (3.4.2.2) ((3.4.5.2)) is readily computable, then the trade-off between cost, level of output attainable and the confidence concerning such attainability may be readily determined via Equations (3.4.2.2), (3.4.5.2) or even (3.4.6). This certainty should facilitate the overall planning of production under uncertainty.

However, since the radial scaling need not be an appropriate scaling operation on inputs (due to changes in effectiveness or learning effect and the value of $K_{\frac{u}{\|u\|}}(p, \xi)$ need not be readily available (for an IS structure the confidence index input sets $CL(u, \xi)$ are of different

shapes as ξ varies), the above scheme of production planning may have limited application. These two drawbacks are partially resolved in the following:

<u>(3.4.7) Definition</u>: Given a scaling operation $*$ on an input space X, a function $h : X \to \mathbb{R}_+$ which is scale homogenous (with $*$) is called a *price function* on X, and $h(x)$ is the *cost* of employing input x.

Let H be the collection of price functions on an input space X with respect to the scaling operation $*$. Modify the definition (3.4.1) of the minimal cost function to:

$$(3.4.8)\quad (u,h,\xi) \in U \times H \times [0,1] \to \tilde{Q}(u,h,\xi) := \begin{cases} \operatorname{Inf}\ \{h(x) \mid x \in CL(u,\xi)\}\ , \\ \quad \text{if}\ \ CL(u,\xi) \neq \emptyset\ ; \\ +\infty\ ,\ \text{if otherwise.} \end{cases}$$

The following straightforward proposition, which also serves to summarize the previous discussion, is valid:

<u>(3.4.9) Proposition</u>: If an input correspondence $L : U \times S \to 2(X)$ has an IS (SS) structure, then its minimal cost function $\tilde{Q}$ is separable in the sense of (3.4.2.1) and (3.4.2.2); ((3.4.4.1), (3.4.5.1) and (3.4.5.2)).

As for the computation of the factor $K_{\frac{u}{\|u\|}}(p,\xi)$ in Equation (3.4.2.2), the following structure is of interest:

<u>(3.4.10) $L : U \times S \to 2(X)$ Has Both a PSS and IS Structure</u>: Recall the definition of a PSS structure (3.3.16). Let $\{S^j\ ;\ j \in J\}$ be the partition of S relevant to the PSS structured input correspondence. Let $\mu^j := \mathbb{P}(S^j)$ and $\mathbb{P}_j(\cdot)$ be the conditional probability measure given

S^j, $j \in J$. For simplicity, assume $\mathcal{P}(S^j) > 0$ for all $j \in J$. Then $\mathcal{P}(\cdot \mid S^j)$ is well defined.

Consider an arbitrary $j \in J$. Define (as done once before) the effective domain $DU^j := \left\{ \frac{u}{\|u\|} \in \Gamma U \mid L^j\left(\frac{u}{\|u\|}, s\right) \neq \emptyset \text{ for some } s \in S^j \right\}$. Since the correspondence L^j is of both IS and SS structure on S^j, it is easy to verify (using representation propositions (3.3.10) and (3.3.11)) that for all $\frac{u}{\|u\|} \in DU^j$, there is a nontrivial homogenous function $\phi^j\left(\cdot, \frac{u}{\|u\|}\right) : X \to \mathbb{R}_+$, a function $G\left(\cdot, \frac{u}{\|u\|}\right) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (3.3.9-i, ii, iii) and scalars $\gamma^j\left(s, \tilde{s}, \frac{u}{\|u\|}\right) \in \mathbb{R}_{++}$ $(s, \tilde{s} \in S^j)$ such that

$$L^j\left(\frac{u}{\|u\|}, \tilde{s}\right) = \left\{ x \in X \mid G\left(\phi^j\left(x, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right) \geq 1 \right\} ; \text{ and}$$

$$L^j\left(\frac{u}{\|u\|}, s\right) = \gamma^j\left(s, \tilde{s}, \frac{u}{\|u\|}\right) \cdot L^j\left(\frac{u}{\|u\|}, \tilde{s}\right), \quad s \in S^j$$

where the state $\tilde{s}$ is taken to be fixed as a reference. By an argument similar to the discussion leading to (3.3.14), $\gamma^j\left(s, \tilde{s}, \frac{u}{\|u\|}\right)$ as a function of s is seen to be measurable on S^j. Then by invoking the homogeneity of $\phi^j\left(\cdot, \frac{u}{\|u\|}\right)$ and letting $\tilde{\gamma}^j\left(s, \frac{u}{\|u\|}\right) := \dfrac{\gamma^j\left(s, \tilde{s}, \frac{u}{\|u\|}\right)}{G^{-1}\left(1, \frac{u}{\|u\|}\right)}$, the input sets $L^j\left(\frac{u}{\|u\|}, s\right)$ has the simple representation

$$L^j\left(\frac{u}{\|u\|}, s\right) = \tilde{\gamma}^j\left(s, \frac{u}{\|u\|}\right) \cdot \left\{ x \in X \mid \phi^j\left(x, \frac{u}{\|u\|}\right) \geq 1 \right\}, \quad s \in S^j, \quad \frac{u}{\|u\|} \in DU^j .$$

For $\frac{u}{\|u\|} \in DU^j$, let $W^j_{\frac{u}{\|u\|}}$ be the conditional distribution function of the random variable $\tilde{\gamma}^j\left(s, \frac{u}{\|u\|}\right)$, i.e.,

$$W^j_{\frac{u}{\|u\|}}(\beta) := \mathcal{P}_j\left\{ s \in S^j \mid \tilde{\gamma}^j\left(s, \frac{u}{\|u\|}\right) \leq \beta \right\}, \quad \beta \in \mathbb{R}_+ .$$

Repeat the above for each $j \in J$.

Now consider an arbitrary $\frac{u}{\|u\|} \in \Gamma U$ and $\xi \in (0,1]$. Denote $J\left(\frac{u}{\|u\|}\right) := \left\{j \in J \mid \frac{u}{\|u\|} \in DU^j\right\}$. One has

$$
\begin{aligned}
CL\left(\frac{u}{\|u\|}, \xi\right) &= \left\{x \in X \mid \mathcal{P}\left\{s \in S \mid x \in L\left(\frac{u}{\|u\|}, s\right)\right\} \geq \xi\right\} \\
&= \left\{x \in X \mid \sum_{j \in J\left(\frac{u}{\|u\|}\right)} \mu^j \cdot \mathcal{P}_j\left\{s \in S^j \mid x \in L^j\left(\frac{u}{\|u\|}, s\right)\right\} \geq \xi\right\} \\
&= \left\{x \in X \mid \sum_{j \in J\left(\frac{u}{\|u\|}\right)} \mu^j \cdot \mathcal{P}_j\left\{s \in S^j \mid \phi^j\left(x, \frac{u}{\|u\|}\right) \geq \tilde{\gamma}^j\left(s, \frac{u}{\|u\|}\right)\right\} \geq \xi\right\} \\
&= \left\{x \in X \mid \sum_{j \in J\left(\frac{u}{\|u\|}\right)} \mu^j \cdot W^j_{\frac{u}{\|u\|}}\left(\phi^j\left(x, \frac{u}{\|u\|}\right)\right) \geq \xi\right\} .
\end{aligned}
$$

Thus $L\left(\frac{u}{\|u\|}, \xi\right)$ takes on the form of a constraint set of a nonlinear mathematical programming problem. Depending on the complexity of the functions $\phi^j\left(\cdot, \frac{u}{\|u\|}\right)$ and the distributions $W^j_{\frac{u}{\|u\|}}$, there is the possibility that the cost factor $K_{\frac{u}{\|u\|}}(p,\xi)$ (see (3.4.4.2)) may be computed as the solution of a mathematical program.

Finally, it is remarked that the slightly more general case of L having only the PSS structure may be handled similarly □

Recall the function Φ defined in (3.3.12). For each $\frac{u}{\|u\|} \in \Gamma U$ and $x \in X$, $\Phi\left(x, \cdot \mid \frac{u}{\|u\|}\right) : S \to \mathbb{R}_+$ represents the maximal output of mix $\frac{u}{\|u\|}$ using input x under the various states of production environments; and the function $\Phi\left(x, \cdot \mid \frac{u}{\|u\|}\right)$ was shown to be measurable.

Given a constraint set C on the input resources, decision scheme (3.4.0-ii) may be formalized as the following problem:

$$\text{For fixed } \frac{u}{\|u\|} \in \Gamma U \text{ , } \operatorname{Max}_x \int_{s \in S} \phi\left(x, s \mid \frac{u}{\|u\|}\right) d\mathcal{P} \tag{3.4.11}$$

$$\text{subject to } x \in C \cap X \text{ .}$$

Problem (3.4.11) may be difficult to solve without assuming special structures on the production technology since the objective function involves an infinite number of random variables, one for each decision variable x . In the following, (3.4.11) is increasingly specialized in several steps with the end-result that it becomes a mathematical programming problem.

<u>Assumption 1</u>: $L : U \times S \to 2(X)$ has a SS structure with a continuous scaling operation $*$. Output mix $\frac{u}{\|u\|} \in DU$.

By the discussion after item (3.3.13), since $\frac{u}{\|u\|} \in DU$, for each $x \in X$, the random variable $\phi\left(x, \cdot \mid \frac{u}{\|u\|}\right) : S \to \mathbb{R}_+$ is distributed according to the distribution function $V\left(\cdot \mid \phi\left(x, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right)$ (see (3.3.14)) where $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ together with the family $\left\{G_s\left(\cdot, \frac{u}{\|u\|}\right) ; s \in S\right\}$ is the representation of L via Proposition (3.3.10). Then problem (3.4.11) may be rewritten as

$$\text{For fixed } \frac{u}{\|u\|} \in DU \text{ , } \operatorname{Max}_x \int_{\mathbb{R}_+} \theta \cdot V\left(d\theta \mid \alpha, \frac{u}{\|u\|}\right)$$

$$\text{subject to } x \in C \cap X \tag{3.4.12}$$

$$\phi\left(x, \frac{u}{\|u\|}\right) - \alpha = 0 \text{ .}$$

There are two difficulties with the solving of problem (3.4.12): (a) the form of the distribution $V\left(\cdot \mid \alpha, \frac{u}{\|u\|}\right)$ in general depends on α ; (b) the computation of $\phi\left(x, \frac{u}{\|u\|}\right)$ may be nontrivial since $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ in general is only scale homogenous. To resolve (a), make

Assumption 2: L is of SSG (3.3.14) structure.

Let $y \in X$ with the expected value $0 < \mathrm{EXP}\left[\phi\left(y \mid \frac{u}{\|u\|}\right)\right] < +\infty$. Tacitly, it is assumed that $V\left(\cdot \mid \alpha, \frac{u}{\|u\|}\right)$ is integrable; for which Axiom P2.I is sufficient. Then by Proposition (3.3.15), for all $x \in S$ and an arbitrary $\tilde{s} \in S$,

$$\mathrm{EXP}\left[\phi\left(x \mid \frac{u}{\|u\|}\right)\right] = \frac{\mathrm{EXP}\left[\phi\left(y \mid \frac{u}{\|u\|}\right)\right]}{G_{\tilde{s}}\left(\phi\left(y \mid \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right)} \cdot G_{\tilde{s}}\left(\phi\left(x \mid \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right).$$

Calling $G_{\tilde{s}}$ simply by G , and letting $B := \mathrm{EXP}\left[\phi\left(y \mid \frac{u}{\|u\|}\right)\right] \Big/ G\left(\phi\left(y, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right)$, (3.4.12) simplifies to

$$\begin{aligned} & \mathrm{Max}_x \quad G\left(\alpha, \tfrac{u}{\|u\|}\right)\cdot B \\ (3.4.13) \qquad & \text{subject to} \quad \phi\left(x, \tfrac{u}{\|u\|}\right) - \alpha = 0 \qquad \left(\tfrac{u}{\|u\|} \in DU\right) \\ & x \in C \cap D \,. \end{aligned}$$

Noting that $G\left(\cdot, \frac{u}{\|u\|}\right)$ is nondecreasing (3.4.9-ii), solving (3.4.13) amounts to solving:

$$(3.4.14) \quad \mathrm{Max}\ \phi\left(x, \frac{u}{\|u\|}\right) \ ; \text{ subject to } \ x \in C \cap D \ ; \ \left(\frac{u}{\|u\|} \in DU\right) .$$

Solving (3.4.14) requires the computation of the scale-homogenous function $\phi\left(\cdot, \frac{u}{\|u\|}\right)$. For this purpose, the property of homogeneity

may be exploited:

(3.4.15) Proposition: Suppose a mapping $F : X \to X$ satisfying (3.2.5-i, ii, iii) and induces a scaling operation $*$ on X via (3.2.5.1). Then if a function $\phi : X \to \mathbb{R}_+$ is scale homogenous (with $*$), the function ω defined by $y \in X \longmapsto \omega(y) := \phi(F(y))$ is homogeneous. Furthermore, for every constraint set $C \subset X$, if y^* solves the problem: $\langle \text{Max } \omega(y) \text{ ; s.t. } y \in X \text{ , } F(y) \subset C \rangle$, then $F(y^*)$ solves the problem: $\langle \text{Max } \phi(x) \text{ ; s.t. } x \in C \cap X \rangle$.

Proof: Since $F(0) = 0$, $\omega(0) = \phi(0) = 0$. For $y \in X$, $y \neq 0$ and $\lambda \in \mathbb{R}_{++}$, by the reversibility of the mapping F and the scale homogeneity of ϕ : $\omega(\lambda \cdot y) = \phi(F(\lambda \cdot y)) = \phi(F(\lambda \cdot \overset{\leftarrow}{F}(F(y)))) = \phi(\lambda * F(y)) = \lambda \cdot (F(y)) = \lambda \cdot \omega(y)$. The proof of the second part of the proposition is just as trivial □

Assumption 3: The scaling operation $*$ relevant for the input correspondence L is induced by a mapping F satisfying (3.2.5-i, ii, iii).

By defining for each $\frac{u}{\|u\|} \in DU$, $y \in X \to \omega\left(y, \frac{u}{\|u\|}\right) := \phi\left(F(y), \frac{u}{\|u\|}\right)$ and applying Proposition (3.4.15), instead of (3.4.14), one may instead choose to solve:

$$\text{(3.4.16)} \quad \max \omega\left(y, \frac{u}{\|u\|}\right) \text{ ; s.t. } y \in X \text{ , } F(y) \in C \text{ ; } \frac{u}{\|u\|} \in DU \text{ .}$$

From the point of view of function forms, problem (3.4.16) need not be easier to solve than (3.4.14). However, recall (from the proof of the representation proposition (3.3.10)) that the function $\phi\left(\cdot, \frac{u}{\|u\|}\right)$ is used to represent the "shape" of the SS structured input correspondence L.

The same reasoning may apply to the purpose of the function $\omega\left(\cdot, \frac{u}{\|u\|}\right)$. This perspective is particularly convincing if L is originally generated by transformation (e.g., see (3.3.7)). Thus, one is primarily interested in the "shape" of the function $\omega\left(\cdot, \frac{u}{\|u\|}\right)$ which may be represented by the subset $\left\{y \in X \mid \omega\left(y, \frac{u}{\|u\|}\right) \geqq 1\right\}$. In many applications, the following assumption is reasonable:

<u>Assumption 4</u>: There exists a finite number of homogenous functions $h_i : X \to \mathbb{R}_+$ $(i = 1, \ldots, N)$ such that the set $\left\{y \in X \mid \omega\left(y, \frac{u}{\|u\|}\right) \geqq 1\right\}$ has the form $\{y \in X \mid h_i(y) \geqq d_i\}$, $d_i > 0$ $(i = 1, \ldots, N)$.

Through Assumption 1 to Assumption 4, the original problem (3.4.11) is seen to be reduced to a mathematical program:

$$
\begin{aligned}
&\text{Max} \quad \beta \\
&\text{subject to} \quad h_i(y) \geqq \beta \cdot d_i \qquad (i = 1, \ldots, N) \\
&\qquad\qquad\quad F(y) \in C \\
&\qquad\qquad\quad y \in X, \ \beta \in \mathbb{R}_+
\end{aligned}
\tag{3.4.17}
$$

as may be easily verified from (3.4.16). Depending on the form of the functions h_i, mapping F and the characterization of the constraint set C, there is the possibility that (3.4.17) may be solved by standard optimization techniques.

The SS structured stochastic input correspondence is admittedly a very special model of production. But its generalization to PSS structures was argued to be a rather reasonable model of technology. To solve (3.4.11) for the case of a PSS structured technology, the sequence of simplification, i.e., Assumption 1 through 4, given above

may be applied to the PSS structure, thus providing a method of solution to an interesting class of technology. In the following, after a short-handed presentation of notations and the underlying reasoning, a mathematical program analogous to (3.4.17) is given for the case of a PSS technology:

$L : U \times S \to 2(X)$ is PSS $\square$ Partition $\{S^j , j \in J\}$ $\square$ $\mu^j := \Phi(S^j)$, conditional probability measure $\Phi_j(\cdot)$ with conditional expectation $EXP_j(\cdot)$ $\square$ $L^j : U \times S^j \to 2(X)$ is SS structured with continuous scaling operation $*_j$ induced by transformation F^j $\square$ Arbitrarily fixed $s^j \in S^j$ $\square$ Assume $\frac{u}{\|u\|} \in DU^j$, $j \in J$ $\square$ $G^j\left(\cdot, \frac{u}{\|u\|}\right) \equiv G_{s^j}\left(\cdot, \frac{u}{\|u\|}\right)$ $\square$ Assume L^j is SSG structured, $j \in J$ $\square$ Representation: $L^j\left(\theta \cdot \frac{u}{\|u\|}, s\right) = \left\{x \mid G_s^j\left(\phi^j\left(x, \frac{u}{\|u\|}\right), \frac{u}{\|u\|}\right) \geqq \theta\right\}$ $\square$ Arbitrarily fix $y^j \in X$, $j \in J$ $\square$ Define $B^j := EXP_j\left[\phi\left(y^j \mid \frac{u}{\|u\|}\right)\right] \Big/ G^j\left(y^j, \frac{u}{\|u\|}\right)$ $\square$ Let $\left\{y \mid \omega^j\left(y, \frac{u}{\|u\|}\right) \geqq 1\right\} = \left\{y \mid \phi^j\left(F^j(y), \frac{u}{\|u\|}\right) \geqq 1\right\} = \{y \in X \mid h_{ji}(y) \geqq d_{ji} , ji = 1, \ldots, N_j\}$, $d_{ji} > 0$.

Then the original problem (3.4.11) is reduced to solving:

$$\text{Max} \sum_{j \in J} B^j \cdot G^j(\beta^j)$$

$$\text{(3.4.18)} \quad \text{subject to} \quad h_{ji}(y) \geqq \beta^j \cdot d_{ji} , \ j \in J , \ i = 1, \ldots, N_j ;$$

$$F^j(y) \subset C , \ j \in J ;$$

$$x \in X , \ \beta^j \in \mathbb{R}_+ .$$

The form of the problem (3.4.18) will be useful as illustration in the next section, which considers the following:

3.5 A Scalar Output Model of Dynamic Production under Uncertainty

A production process is carried out in T periods $(T > 1)$. There is only one product, the *cumulative* output of which over the T periods of production is assumed to come out at the end of the T-th period. There are n inputs, and an input history is represented by a T-tuple $(y_1, \ldots, y_T)$ where $y_t \in \mathbb{R}^n_+$ is the input into the production system at the beginning of the t-th period. Thus the output space U is taken to be $\mathbb{R}_+$ while the input space $X \equiv (l^T_\infty)^n_+$.

The technology of the production process is modelled by an input correspondence $L : U \times S \to 2^{(X)}$ which has a PSSG structure with the usual *radial scaling*. The relevant PSS partition is $\{S^j \ ; \ j = 1, \ldots, J\}$ and J is assumed to be finite. The input correspondence satisfies P2.I in addition to the stochastic weak axioms (1.6.4).

The producer has a (subjective) *probability assessment* $\mathcal{P}$ on the state space $(S, \mathcal{B})$ and an information structure $\mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_T)$. The partition $\mathcal{I}_t$ of S represents the information available at the beginning of the t-th period. For each $s \in S$, $(I_1(s), I_2(s), \ldots, I_T(s))$ denotes the sequence of realized information. A production policy is simply taken to be represented by an input mapping $s \in S \longmapsto \underline{x}(s) \equiv (\underline{x}_1(s), \ldots, \underline{x}_T(s)) \in (l^T_\infty)^n_+$, representing the choice of input histories. A policy $\underline{x}$ is consistent with the information structure if s, $\tilde{s} \in I_t \in \mathcal{I}_t$ implies $\underline{x}_t(s) = \underline{x}_t(\tilde{s})$. (Alternatively, it may be convenient to think of a consistent policy $\underline{x}$ as having $\underline{x}_t(s)$ depending only on $I_t(s)$.) Denote the space of consistent (input) policy by $\mathfrak{X}$. The following assumptions are imposed on the information structure:

(I.1) every $x \in \mathfrak{X}$ is measurable;

(I.2) every $I_t \in \mathcal{I}_t$ is either a singleton or has $\mathcal{P}(I_t) > 0$,

$(t = 1, \ldots, T)$;

(I.3) $\mathcal{J}$ is expanding; i.e., $I_1(s) \supset I_2(s) \supset \cdots \supset I_t(s) \quad (s \in S)$.

Suppose the producer is faced with some resource constraint which is modelled by a compact subset C of X . A consistent policy $\underline{x}$ is admissible if for all $s \in S$, $\underline{x}(s) \in C$.

The consequence of an admissible input policy $\underline{x}$ is the *maximal cumulative* output it yields at the end of period $T : s \in S \longmapsto \Phi(\underline{x}(s),s)$ where the function Φ is that defined in (3.3.12) (with the superfluous argument $\frac{u}{\|u\|}$ dropped since $U \equiv \mathbb{R}_+$).

The problem of the producer is to formulate the best admissible input policy; formally, to solve:

$$\text{(P)} \qquad \begin{array}{c} \text{Sup EXP } [\Phi(\underline{x}(s),s)] \\ \text{subject to } \underline{x} \text{ being an admissible policy.} \end{array}$$

In the following, it is to be shown that (P) is well defined and has an optimal solution. The exposition somewhat parallels the first part of Rockerfeller/Wets' [1976] paper on multi-stage stochastic convex programming. Some preliminary considerations are given first:

It had been shown that for all $y \in X$, the function $s \in S \to \Phi(y,s)$ is measurable. However, it is not clear whether for all consistent policy $\underline{x}$, $s \in S \to \Phi(\underline{x}(s),s)$ is measurable. The following simple fact (proof omitted) is relevant:

<u>(3.5.1) Fact</u>: If $\mathcal{J}_t$ is a countable partition of S for all $t = 1, \ldots, T$; then every consistent policy $\underline{x} \in \mathcal{X}$ has $s \in S \to \Phi(\underline{x}(s),s)$ measurable. Otherwise, recalling the representation of a SS (hence a PSS) structure, if either (i) $\phi^j : X \to \mathbb{R}_+$ is continuous $(j = 1, \ldots, J)$;

or (ii) $y' \geqq y''$ implies $\phi^j(y') \geqq \phi^j(y'')$ $(j = 1, \ldots, J)$; then the same conclusion holds.

In any case, in order for Problem (P) to be meaningful, it is *assumed* that *for all* $\underline{x} \in \mathfrak{X}$, $s \in S \to \Phi(\underline{x}(s), s)$ *is measurable.*

The following notation will be used:

(a) If $z \in (1_\infty^\sigma)_+$ and $\sigma \geqq t$, a projection on the first t components is defined by $\nabla_t z := (z_1, z_2, \ldots, z_t)$; in particular, $y^t := (y_1, \ldots, y_t) = \nabla_t y$ for all $y \in X$ $(t = 1, \ldots, T)$. For the constraint set C , $C^t := \nabla_t C = \{y \in (1_\infty^t)_+^n \mid y = \nabla_t z \ , \ z \in C\}$, $t = 1, \ldots, T$.

(b) $\underline{x}^t(s) := (\underline{x}_1(s), \ldots, \underline{x}_t(s))$, $s \in S$, $\underline{x} \in \mathfrak{X}$ $(t = 1, \ldots, T)$.

(c) $\underline{x}^t \in C^t$ iff $\underline{x}^t(s) \in C^t$ for all $s \in S$ $(t = 1, \ldots, T)$.

(d) $\underline{x}^t \in \mathfrak{X}^t$ if $\underline{x}^t(\tilde{s}) = \underline{x}^t(s)$ for all s , $\tilde{s} \in I_t \in \mathcal{J}_t$ $(t = 1, \ldots, T)$.

(e) For $\underline{x}^t \in \mathfrak{X}^t$ and $I_\sigma \in \mathcal{J}_\sigma$ $(\sigma \leqq t)$, $\underline{x}^\sigma(I_\sigma)$ denotes the constant value $\underline{x}^\sigma$ takes on the set I_σ .

The following simple facts (proof omitted) and lemma are useful:

<u>(3.5.2) Fact</u>: Let C be a compact subset of $(1_\infty^T)_+^n$. For each $t = 1, \ldots, T-1$, the correspondence $z \in (1_\infty^t)_+^n \to D^t(z) := \{y \in (1_\infty^T)_+^n \mid y \in C \ , \ y^t = z\}$ is compact-valued and upper-hemi-continuous.

<u>(3.5.3) Fact</u>: Suppose $f : (1_\infty^T)_+^n \to \mathbb{R}$ is upper-semi-continuous. For the correspondences D^t $(t = 1, \ldots, T-1)$ defined above, let $\mathcal{D}^t := \{z \in (1_\infty^t)_+^n \mid D^t(z) \neq \emptyset\}$. Then the function

$$z \in \mathcal{D}^t \to \mathrm{Max}\ \{f(y) \mid y \in D^t(z)\}\ ,\ t = 1, \ldots, T\ ;$$

is well defined and u.s.c. on its domain.

(3.5.4) Lemma: With the assumption of P2.I on the technology, there is a function $g : S \to \mathbb{R}_+$ which is integrable such that for all $\underline{x} \in \mathfrak{X} \cap C$, $\Phi(\underline{x}(s),s) \leqq g(s)$ for all $s \in S$.

Proof: Since the technology has a PSS structure, it has the representation: $L^j(u,s) = \{y \in X \mid G_s^j(\phi^j(y)) \geqq 1\}$, $s \in S^j$ $(j = 1, \ldots, J)$. Since the functions ϕ^j are u.s.c. and C is compact, one may choose an input y^j for each $j = 1, \ldots, J$ such that $\phi^j(y^j) = \text{Max}\,\{\phi^j(y) \mid y \in C\}$. Then by the choice of y^j, for every $\underline{x} \in \mathfrak{X} \cap C$, $\Phi(\underline{x}(s),s) \leqq G_s^j(\phi^j(y^j))$ if $s \in S^j$ since G_s^j is nondecreasing. Then it follows that for all $\underline{x} \in \mathfrak{X} \cap C$ and $s \in S$, $\Phi(\underline{x}(s),s) \leqq \underset{j=1,\ldots,J}{\text{Max}} \{G_s^j(\phi^j(y^j))\}$. Because Property P2.I is assumed to hold, for each of the y^j chosen, there is an integrable function $g^j : S \to \mathbb{R}_+$ such that $G_s^j(\phi^j(y^j)) \leqq g^j(s)$ for all $s \in S$. The proof is completed by letting $g(s) := \underset{j=1,\ldots,J}{\text{Max}} \{g^j(s)\}$ □

The solution of the problem (P) is related to a "backward" dynamic programming problem developed below. Define for the T-th period:

$$(y^T, I_T) \in (1_\infty^T)_+^n \times \mathcal{J}_T \to Q^T(y^T, I_T) := \text{EXP}\,[\Phi(y,s) \mid I_T] .$$

By Assumption (I.2) on the information structure, and P.2 (or P2.I), it is clear that Q^T is well defined and finite. Moreover, one has:

(3.5.5) Lemma: For each $I_T \in \mathcal{J}_T$, $Q^T(\cdot, I_T)$ is u.s.c.

Proof: Consider an arbitrary $I_T \in \mathcal{J}_T$ with $\mathcal{P}(I_T) > 0$. Let I_T be partitioned into $\{I_T \cap S^j ; j = 1, \ldots, J\}$ where $\{S^j ; j = 1, \ldots, J\}$

is the partition of the PSSG technology. Denote $J(I_T) :=$ $\{j \mid \mathcal{P}(I_T \cap S^j \mid I_T) > 0\}$. Then for every $y \in (l_\infty^T)_+^n$

$$(3.5.5.1) \quad Q^T(y,I_T) = \sum_{j \in J(I_T)} \text{EXP}\,[\phi(y,s) \mid I_T \cap S^j] \cdot \mathcal{P}(I_T \cap S^j \mid I_T) .$$

For each $j \in J(I_T)$, by Proposition (3.3.15) as applied to the event $(I_T \cap S^j)$ with the attendant conditional probability, there exists a positive scalar B^j and a pseudo-production function G^j such that

$$(3.5.5.2) \quad \text{EXP}\,[\phi(y,s) \mid I_T \cap S^j] = B^j \cdot G^j(\phi^j(y)) \; , \; j \in J(I_T) .$$

Since ϕ^j is u.s.c. and G^j is nondecreasing, the function $G^j(\phi^j(\cdot))$ is u.s.c. for all $j \in J(I_T)$. Since there are only a finite number of indices in $J(I_T)$, for the fixed I_T , the mapping $Q^T(\cdot,I_T)$ is u.s.c. on $(l_\infty^T)_+^n$.

The simpler case of I_T being a singleton may be treated analogously □

Now, define for $t = 1, \ldots, T-1$

$$(y^t,I_t) \in C^t \times \mathcal{I}_t \to Q^t(y^t,I_t) := \text{EXP}\left\{ \begin{array}{l} \text{Max} \;\; Q^{t+1}(y^{t+1},I_{t+1}) \\ \text{s.t.} \;\; \nabla_t y^{t+1} = y^t \\ \qquad\quad y^{t+1} \in C^{t+1} \end{array} \;\middle|\; I_t \right\} .$$

<u>(3.5.6) Lemma</u>: For each $t = 1, \ldots, T-1$, Q^t is well defined; and for every $I_t \in \mathcal{I}_T$, $Q^t(\cdot,I_t)$ is u.s.c. on C^t .

<u>Proof</u>: Consider $t = T-1$. First note that if $\tilde{y}^{T-1} \in C^{T-1}$, then the constraint set $\{y^T \mid y^T \in C^T , \nabla_{T-1} y^T = \tilde{y}^{T-1}\}$ is not empty.

Since for all $I_T \in \mathcal{J}_T$, $Q^T(\cdot, I_T)$ is u.s.c. on $(1^T_{=})^n_+$ and $C^T \equiv C$ is compact, the optimal solution

$$f(\tilde{y}^{T-1}, I_T) := \text{Max } Q^T(y^T, I_T)$$

$$\text{subject to } \nabla_{T-1} y^T = \tilde{y}^{T-1}, \ y^T \in C^T \quad (\tilde{y}^{T-1} \in C^{T-1}, \ I_T \in \mathcal{J}_T)$$

is well defined. Furthermore, by Fact (3.5.3), $f(\cdot, I_T)$ is u.s.c. on C^{T-1} for all $I_T \in \mathcal{J}_T$. Now it is to be shown that $Q^{T-1}(\cdot, I_{T-1})$ is u.s.c. on C^{T-1} for all $I_{T-1} \in \mathcal{J}_{T-1}$. For the case that I_{T-1} is a singleton, $Q^{T-1}(\cdot, I_{T-1})$ is clearly u.s.c. since it is precisely $f(\cdot, I_T \equiv I_{T-1})$ according to the assumption of expanding information. So suppose $I_{T-1} \in \mathcal{J}_{T-1}$ is not a singleton. Since then $\mathcal{P}(I_{T-1}) > 0$, the conditional probability and expectation given I_{T-1} is well defined.

Let $\{y^k\} \subset C^{T-1}$ converges to $y^o \in C^{T-1}$.

Recall the lemma (3.5.4). Define for $s \in I_T \in \mathcal{J}_T$:- $h(s) := \text{EXP}\ [g(s) \mid I_T]$. Clearly the function h thus defined is integrable. Furthermore, for each $y^{T-1} \in C^{T-1}$, $h(s) \geq f(y^{T-1}, I_T)$ for $s \in I_T$ $(I_T \in \mathcal{J}_T)$. Then

$$\begin{aligned}
\limsup_k Q^{T-1}(y^k, I_{T-1}) &= \limsup_k \int_{I_T \subset I_{T-1}} f(y^k, I_T) \cdot \mathcal{P}(I_T \mid I_{T-1}) \\
&\leq \int_{I_T \subset I_{T-1}} \limsup_k f(y^k, I_T) \cdot \mathcal{P}(I_T \mid I_{T-1}) \\
&\leq \int_{I_T \subset I_{T-1}} f(y^o, I_T) \cdot \mathcal{P}(I_T \mid I_{T-1}) \\
&= Q^{T-1}(y^o, I_{T-1}) .
\end{aligned}$$

This establishes the u.s.c. of $Q^{T-1}(\cdot, I_{T-1})$ on C^{T-1}; $(I_{T-1} \in \mathcal{J}_{T-1})$.

By repeating the above argument for $t = T-2, T-1, \ldots, 2, 1$; the lemma is established □

<u>(3.5.7) Proposition</u>: Suppose $\underline{x}^* \in \mathfrak{X}$ solves the problem

$$\text{(P)} \qquad \text{Max EXP } [\phi(\underline{x}(s),s)] \text{ subject to } \underline{x} \in \mathfrak{X} \cap C ,$$

then for all $I_t \in \mathcal{J}_t$ with $\mathcal{P}(I_t) > 0$, $t = 1, \ldots, T$, $\underline{x}^{*t}(I_t)$ solves the problem

$$(P^t, I_t) \qquad \text{Max } Q^t(y^t, I_t) \text{ subject to } y^t \in C^t .$$

Conversely, if $\underline{x}^{*t} \in \mathfrak{X}^t \cap C^t$ $(t \leq T)$ has $\underline{x}^{*t}(I_t)$ solve (P^t, I_t) for all $I_t \in \mathcal{J}_t$; then $\underline{x}^{*t}$ may be "extended" to a solution of (P).

<u>Proof</u>: The proof will be carried out for $T = 2$. The reasoning used applies to the general case.

For a fixed $\bar{y}^1 \in C^1$, for each $I_1 \in \mathcal{J}_1$

$$\text{(1)} \quad Q^1(\bar{y}^1, I_1) \equiv \text{EXP } [\text{Max } Q^2(y^2, I_2) \text{ ; s.t. } \nabla_1 y^2 = \bar{y}^1 \text{ , } y^2 \in C^2 = C \mid I_1]$$

$$\text{(2)} \qquad = \text{EXP } [\text{Max EXP } [\phi(y^2, s) \mid I_2] \text{ ; s.t. } \nabla_1 y^2 = \bar{y}^1 \text{ , } y^2 \in C \mid I_1]$$

$$\text{(3)} \qquad = \text{Max EXP } [\phi(\underline{x}(s), s) \mid I_1]$$

$$\text{s.t. } \nabla_1 \underline{x}(s) = \bar{y}^1 \text{ and } \underline{x}(s) \in C \text{ for } s \in I_1 \text{ ;}$$

$$s \text{ , } \bar{s} \in I_2 \subset I_1 \text{ implies } \underline{x}(s) = \underline{x}(\bar{s}) .$$

The above string of equalities is well defined according to Lemma (3.5.6) and (3.5.5). Note that the optimal solution $\underline{x}$ of (3) may differ from that of (2) on I_2 $(\subset I_1)$ if $\mathcal{P}(I_2) = 0$ and $\mathcal{P}(I_1) > 0$. It follows from above that

$$\text{(4)}\quad \begin{array}{ll} \operatorname{Max} Q^1(y^1, I_1) = & \operatorname{Max} \operatorname{EXP}[\Phi(\underline{x}(s), s) \mid I_1] \\ \text{s.t. } y^1 \in C^1 & \text{s.t. } \underline{x}(s) \in C \text{ for } s \in I_1 \\ & s, \tilde{s} \in I_2 \subset I_1 \text{ implies } \underline{x}(s) = \underline{x}(\tilde{s}) . \end{array}$$

Suppose $\underline{x}^* \in \mathfrak{X} \cap C$ solves (P). Consider an $I_1 \in \mathcal{J}_1$ with $(I_1) > 0$. Let $\underline{z}$ be another admissible policy, i.e., $\underline{z} \in \mathfrak{X} \cap C$, then it must be true that $\operatorname{EXP}[\Phi(\underline{x}^*(s), s) \mid I_1] \geq \operatorname{EXP}[\Phi(\underline{z}(s), s) \mid I_1]$; since if otherwise, a simple modification of $\underline{x}^*$ will lead to a contradiction with the optimality of $\underline{x}^*$. Hence $\underline{x}^{*1}(I_1)$ solves (P^1, I_1) . The argument needed for the case of I_1 with $\mathcal{P}(I_1) = 0$ is just as trivial.

Conversely, suppose $\underline{x}^{*1} \in \mathfrak{X}^1 \cap C^1$ has $\underline{x}^{*1}(I_1)$ solving (P^1, I_1) for all $I_1 \in \mathcal{J}_1$. Then for every $I_1 \in \mathcal{J}_1$, there is a mapping $\underline{z} : I_1 \to C^2$ such that $\underline{z}^1(s) = \underline{x}^{*1}(I_1)$ for all $s \in I_1$, and $s , \tilde{s} \in I_2$ $(I_2 \subset I_1)$ implies $\underline{z}(s) = \underline{z}(\tilde{s})$; and $\underline{z}$ is an optimal solution to (3). The extension of $\underline{x}^{*1} \in \mathfrak{X}^1 \cap C^1$ above clearly results in a policy $\underline{z} \in \mathfrak{X} \cap C$. Since $\underline{z}$ solves (4) on every $I_1 \in \mathcal{J}_1$, $\underline{z}$ solves the original problem (P) □

Since $Q^t(\cdot, I_t)$ is u.s.c. on C^t $(t = 1, \ldots, T)$, the problem (P^t, I_t) is solvable for all $I_t \in \mathcal{J}_t$. Thus, by the second part of the proposition, there exists an optimal solution to (P). Furthermore, the above proposition formalizes the intuition that a production policy $\underline{x} \in \mathfrak{X} \cap C$ is optimal if and only if the input decision at each time t , based on the information I_t and the earlier decisions, is optimal almost surely.

To find the optimal solution to Problem (P), the dynamic programs (P^t, I_t) in terms of the value functions Q^t may not be helpful since in general Q^t is not known, and not easily computable. However, with some further assumptions, the PSSG production structure does allow for a computation of the optimal policy via the reduction process given in (3.4.11) through (3.4.18). As an illustration, consider:

<u>(3.5.8) Example</u>: Assume the partition $\mathcal{J}_T$ *is a finite partition.* Since the information structure is expanding, each state $s \in S$ is associated with a unique information sequence $(I_1(s), \ldots, I_T(s))$. Since $\mathcal{J}_T$ is finite, there is at most a finite number of such information sequences. Denote the collection of information sequences by IP (paths of information). A consistent policy $\underline{x} \in \mathfrak{X}$ may be thought of as an input mapping $\underline{x} : IP \to X$ such that $I \in IP \to \underline{x}(I_1, \ldots, I_T) = (\underline{x}_1(I_1), \ldots, \underline{x}_T(I_T))$ where the input $\underline{x}_t$ at time t depends only on I_t , $t = 1, \ldots, T$.

Assume $\mathcal{J}_T$ *is finer than* $\{S^j , j = 1, \ldots, J\}$. That is, at the last period T , the qualitative classification of the prevailing input correspondence is known. The remaining uncertainty only concerns the scaling law of production. Given a realization $I = (I_1, \ldots, I_T)$, the prevailing input correspondence will be indexed by $j(I)$. As in Equation (3.5.5.2), the conditional expectation of the output attainable given I for a consistent policy $x \in \mathfrak{X}$ is:

$$\text{EXP}\ [\phi(\underline{x}(s),s) \mid I] = B^I \cdot G^{j(I)}(\phi^{j(I)}(\underline{x}(I)))$$

where G^j's are the appropriate scalar-valued production function, and the factors B^i depends on I_T .

Then the problem (P) may be written as

$$\text{(P')} \quad \max \sum_{I \in IP} B^{I} \cdot G^{j(I)}(\phi^{j(I)}(\underline{x}(I))) \cdot (I) \ ; \text{ s.t. } \ \underline{x}(I) \in C \ , \ \underline{x} \in \mathfrak{X} \ .$$

Following the reduction exhibited by Assumption 4 and (3.4.18), an interesting case arises when (i) the technology is of linear activity analysis type; i.e., for some appropriate matrices A^j and vectors d^j, $\{y \in X \mid \phi^j(y) \geqq 1\} \equiv \{y \in X \mid A^j \cdot y \geqq d^j\}$, $j = 1, \ldots, J$; (ii) the constraint set C is given by $C \equiv \{y \in X \mid D \cdot y \leqq e\}$; and (iii) constant return to scale prevails, i.e., $G^j(\alpha) = k^j \cdot \alpha$ for some $k^j \in \mathbb{R}_{++}$, $j = 1, \ldots, J$. Then (P) is reduced to a (possibly large scale) special structured linear programming problem

$$\begin{aligned} \max \ & \sum_{I \in IP} \sigma^{I} \cdot \beta^{I} \\ \text{s.t. } \ & A^{j(I)} \cdot \underline{x}(I) \geqq \beta^{I} \cdot d^{j(I)} \\ & D \cdot \underline{x}(I) \leqq e \\ & \beta^{I} \in \mathbb{R}_{+} \ , \ I \in IP \ ; \ \underline{x} \in \mathfrak{X} \end{aligned}$$

where for $I \in IP$, $\sigma^{I} := B^{I} \cdot \mathcal{P}(I)$ with B^{I} an appropriately chosen weighting factor (see (3.4.13)).

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